Department of Physics and Astronomy, University of Canterbury,
Private Bag 4800, Christchurch, New Zealand

Hamiltonian Gauge Systems

With application to General Relativity and Shape Dynamics

Hadleigh Frost

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Supervisor: Prof. D. L. Wiltshire

Abstract

In this report we analyse the Hamiltonian formulation of guage theories and explore the consequences of this analysis for electromagnetism, Hamiltonian general relativity, and shape dynamics. It is demonstrated that the Dirac conjecture, and several of its corollaries, are incorrect. However, we prove weaker versions of these results. An alternative to the Dirac conjecture, advanced by Pons, is considered. We give a more complete proof of Pons's result that includes an additional case.

These results are applied to the case of Hamiltonian general relativity. A formulation of Hamiltonian general relativity is given that includes the lapse scalar and shift vector as canonical variables. It is found that the gauge freedom of Hamiltonian general relativity only agrees with standard general relativity if the lapse and shift are included in this way. In related work, shape dynamics is also investigated. We reproduce the construction of shape dynamics for a closed manifold and investigate the gauge freedom of this theory without using the Dirac conjecture.

Contents

Chapter 1

Introduction

General relativity is a theory of gravity that describes distances and durations using a 4-dimensional curved Lorentzian manifold called spacetime. It has been well tested at the scale of the solar system and has been successfully employed to describe the cosmological history of the universe as well as astronomical phenomena such as gravitational lensing. However, general relativity ceases to be valid in the strong gravity regime realised by black holes and the big bang singularity. In these limits, quantum effects become important, but standard methods of quantisation cannot be easily applied to general relativity. This is because general relativity is a theory of spacetime itself, as opposed to a theory of degrees of freedom on an ambient, background spacetime. In particular, conventional notions of time in quantum mechanics (such as unitary time-evolution generated by the Hamiltonian) cannot be easily reconciled with general relativity in which time is a general coordinate of a 4-dimensional, diffeomorphism-invariant manifold. This is called *the problem of time* (see [1] for a review).

At the classical level, one can introduce a global time function which breaks spacetime into a foliation of spatial hypersurfaces. This is called the '3+1'-split in the literature, and we note that it is not possible, in general, to perform this decomposition of spacetime. In particular, the existence of such a foliation is not possible in spacetimes that are not globally hyperbolic: that is, spacetimes in which it is impossible to construct a hypersurface intersected precisely once by every non-spacelike curve [2]. Nevertheless, assuming such a '3+1'-split, one can reformulate general relativity as a Hamiltonian theory of a 3-dimensional spatial metric and two additional quantities: a scalar field called *the lapse*, and a 3-dimensional vector field called *the shift*. This approach was pioneered by Arnowitt, Deser and Misner (ADM) [3]. Despite the ADM approach admitting a Hamiltonian formulation, this does not resolve the problem of time. This is because the theory should not depend on the particular foliation we choose: this is called refoliation invariance. However, each foliation corresponds to a different choice of time which makes the quantisation procedure ambiguous. This is called *the many-fingered problem of time*.

Shape Dynamics

Many extensions of general relativity (GR) have been proposed to include conformal invariance in four dimensions. Notably, a whole class of such theories called *scalar-tensor theories* have been extensively investigated [4]. However, these theories make physical predictions that differ from GR and they have been largely ruled out by experiment. Shape dynamics is a proposed theory of gravity that maintains physical equivalence with GR whilst also exhibiting invariance under conformal transformations (in a 3-dimensional sense that is explained in Chapter 4). The theory was first proposed in 2010 and early results suggest that it may be useful for advancing the program of quantum gravity as well as theoretical cosmology. In particular, a correspondence between gravity and conformal field theories (CFTs) emerges quite easily from shape dynamics [5, 6]. Moreover, shape dynamics does not exhibit the problem of many-fingered time problem — although, the quantisation of shape dynamics is hindered by other difficulties [7].

Shape dynamics gets its name from Barbour's program of implementing the Machian principle of *relationism* in mechanics (e.g. [8]). I will omit the details of the long, attendant philosophical discussion about the nature of mechanics (see [9] for an 'Aristotle to Einstein' review). Moreover, if the reader is interested in the variety of recent perspectives on relational ideas, they are referred to Pooley's review [10] (as well as the excellent bibliography contained in Anderson's essay [11]). However, the idea is essentially that we can only define distances and durations relative to other distances and durations. Consider, for instance, a collection of particles. In lieu of an externally opposed ruler, the distances between the particles can only be determined by comparing the separations of the particles. That is, by looking at the *shape* of the configuration. Thus, according to the relativist, any physical description of the particles should be invariant under global dilatations since these do not change the shape of the particles. Moreover, in lieu of an externally imposed clock, time should be abstracted (somehow) from physical changes in the configuration of the particles.

It is perhaps surprising that these ideas can be realised in a formal mathematical description of Newtonian mechanics. Beginning in the late seventies, Barbour and Bertotti studied formulations of the few-body Newtonian problems in the relational, or 'shape' configuration space [12, 13]. More recently, this approach has given some interesting results. For instance, for the Newtonian few-body problem, Barbour and collaborators showed that time evolution through shape space is asymmetric in time: the 'complexity' of the system increases only in one direction of time [14, 15]. This might suggest an origin of the arrow of time. Further, Koslowski studied a similar system and showed that the dynamics exhibits an inflation-type behaviour when described using shape space [16].

General relativity is not Newtonian mechanics and Barbour's ideas resist being applied to GR. Nevertheless, since shape dynamics exhibits 3-dimensional conformal invariance, the shape dynamics configuration space is ignorant of scale and only contains shape degrees of freedom — much like Barbour's simpler models. It is for this reason that shape dynamics was given this name.

In practical terms, shape dynamics is constructed from Hamiltonian general relativity. Starting with Hamiltonian GR, one attempts to replace refoliation invariance with a different gauge freedom: invariance under (spatial, volume preserving) conformal transformations. This was first achieved for the case of a closed universe in reference [17] using an approach inspired, in part, by the Stuekelberg extension from field theory. The same authors later proposed a cleaner version of this approach which they call the *linking theory* construction. That work rests heavily on the Hamiltonian description of gauge systems, which concerns a good part of this report.

Hamiltonian Gauge Systems

A theory is said to possess *gauge freedom* if the theory gives a redundant description of physical states. That is, for any given physical state, one has the freedom to choose a representative of that state in one's theory. Correspondingly, a transformation of the theory that does not alter the physical state described by the theory is called a *gauge transformation*. In the Hamiltonian formalism, gauge theories become constrained Hamiltonian systems.

(The reason for this is described in Chapter 2.) Starting with Dirac's analysis in the 1930s [18], working physicists typically study such constrained Hamiltonian systems using what is now known as the Dirac-Bergmann formalism. This approach emphasises the structure of the algebra formed by the constraints of the theory (see [19]). Unfortunately, it happens that this approach is flawed and it fails to give sensible results for the Hamiltonian theory of electromagnetism, for instance.

The flawed Dirac-Bergmann approach may also have consequences for gravity research. As discussed above, GR can (in some cases) be formulated as a Hamiltonian theory. Since GR has gauge freedom, the corresponding Hamiltonian formulation is a constrained system. Motivated, in part, by the habits of the Dirac-Bergmann approach to such systems, physicists often claim that Hamiltonian GR is a theory only of the spatial 3-metric, while ignoring the lapse and shift. (See Wald [20], and also REFS.)

In spite of this surprisingly confused literature on the subject, there are some alternative approaches to Hamiltonian gauge systems. Faddeev and Jackiw gave a novel algorithm for gauge fixing Hamiltonian systems [21]. Their algorithm relies solely on variational principles and systematically modifies the action until the theory no-longer has any gauge freedom. On the other hand, work by Pons, Shepley, and others attempts to directly correct the problems in the Dirac-Bergmann approach [22, 23, 24]. It is this work that we build on this report.

Outline

This report gives an analysis of classical Hamiltonian gauge theories and explores the consequences of this analysis for electromagnetism, Hamiltonian GR and shape dynamics. In Chapter 2, Hamiltonian gauge systems are discussed and a consistent formalism is developed. An elementary discussion of the geometry of the Legendre transformation is given and, in apparently original work, it is demonstrated that the Dirac consistency algorithm always terminates. It is further argued that Dirac's analysis of gauge transformations is incorrect and electromagnetism is given as a counter-example. As a consequence of this analysis, a number of problems are identified in the analysis of Henneaux and Teitelboim. However, in original work, weaker versions of two of Henneaux and Teitelboims' results are proven.

Chapter 2 also extends the work of Pons and collaborators. A more complete proof of Pons's characterisation of gauge generators is given that considers rigid, time-independent gauge transformations in addition to gauge freedom arising from the equations of motion. These results are applied to the Hamiltonian theory of electromagnetism and a complete analysis of the theory's gauge freedom is given.

The most important original result concerns Hamiltonian GR and is discussed in Chapter 3. A derivation of Hamiltonian GR is given that retains the lapse function and shift vector as variables in the phase space of the theory. An analysis of the gauge freedom of Hamiltonian GR is given using the results in Chapter 2. This analysis shows that including the shift changes the gauge freedom of the theory and, consequently, it is argued that the shift should not be discarded from the theory as is done by many authors. It is not immediately clear, however, what implications this has for quantisation.

Finally, shape dynamics is discussed in Chapter 4. Shape dynamics is constructed for the case of a closed universe and the technical calculations attendant to this construction are given in an appendix to this report. Moreover, the derivation of shape dynamics presented here does not use the Dirac-Bergman language and adopts the approach of Chapter 2 instead. Despite this, the main results from the literature are reproduced and an analysis of the gauge freedoms of shape dynamics confirms that it possesses 3-dimensional conformal invariance.

Chapter 2

Hamiltonian Gauge Systems

Consider a system whose possible configurations are described by an N-dimensional space $\mathcal{Q}$ and whose dynamics are described by the extremals of

$$S[\mathbf{q}] = \int \mathrm{d}t \; L(\mathbf{q}, \dot{\mathbf{q}}), \tag{2.1}$$

for some Lagrangian function $L(\mathbf{q}, \dot{\mathbf{q}})$. By taking variations of the action one can show that the extremals $\mathbf{q}(t)$ satisfy the equations of motion

$$\frac{\partial^2 L}{\partial \dot{q}_n \partial \dot{q}_m} \ddot{q}_m + \frac{\partial^2 L}{\partial \dot{q}_n \partial \dot{q}_m} \dot{q}_m - \frac{\partial L}{\partial q_n} = 0. \tag{2.2}$$

So, if we choose some initial data $\mathbf{q}_0 \in \mathcal{Q}$ and $\dot{\mathbf{q}}_0 \in T_{\mathbf{q}_0}\mathcal{Q}$ the subsequent time evolution of the system is uniquely determined by (2.2) if and only if the Hessian matrix,

$$W_{mn} = \frac{\partial^2 L}{\partial \dot{q}_n \partial \dot{q}_m}, \tag{2.3}$$

is invertible. When the Hessian is not invertible the equations of motion allow for many possible paths $\mathbf{q}(t)$ that satisfy the initial data. That is, our description of the physical system using $T\mathcal{Q}$ is degenerate: multiple points in $T\mathcal{Q}$ correspond to the same physical state. Such a theory is said to have gauge freedom because we are free to choose which points in $T\mathcal{Q}$ to represent each of the physical states. Let us call this type of gauge freedom — that is manifest in the equations of motion — *dynamical gauge freedom.* In this chapter we will investigate theories with dynamical gauge freedom in the canonical formalism. Of course, theories may also have *non-dynamical gauge freedom*; that is, a redundancy in $T\mathcal{Q}$ that is not apparent from the equations of motion. For example, consider a particle moving in one dimension in a linear potential. The equations of motion for the particle are completely deterministic, but we have the freedom to choose the position of the origin. Thus, the system has a non-dynamical gauge freedom ($x \mapsto x + a$) on the initial data but no dynamical gauge freedom. With this distinction in mind, let us begin our discussion of the canonical formalism.

2.1 The Canonical Formalism

Let us describe a theory with dynamical gauge freedom using the Hamiltonian formalism. Given the Lagrangian description on $T\mathcal{Q}$, we obtain the Hamiltonian description by performing the Legendre transformation $F : T\mathcal{Q} \to T^*\mathcal{Q}$ that is defined by $\mathbf{q} \mapsto \mathbf{q}$ and $\dot{\mathbf{q}} \mapsto \mathbf{p} = \mathbf{P}(\mathbf{q}, \dot{\mathbf{q}})$ where the momentum functions are

$$P_n(\mathbf{q}, \dot{\mathbf{q}}) = \frac{\partial L}{\partial \dot{q}_n}. \tag{2.4}$$

The Jacobian of this transformation is

$$\frac{\partial P_n}{\partial \dot{q}_m} = \frac{\partial^2 L}{\partial \dot{q}_n \partial \dot{q}_m}, \tag{2.5}$$

which is just the Hessian matrix W_{mn} that we considered above. We are considering a theory with a singular Hessian, and so this Legendre transformation has a non-trivial kernel. Consequently, the image of $T\mathcal{Q}$ under F is some lower-dimensional subspace $F(T\mathcal{Q}) = \mathcal{C}$ of $T^*\mathcal{Q}$, and for every point $\mathbf{x} \in \mathcal{C}$ we have a corresponding subspace $F^{-1}(\mathbf{x})$ in $T\mathcal{Q}$. In this way the Legendre transformation defines a foliation of $T\mathcal{Q}$ into subspaces. Moreover, given a function $A(\mathbf{q}, \dot{\mathbf{q}})$ on $T\mathcal{Q}$ we can only define a corresponding function A^* on $T^*\mathcal{Q}$ if A is constant along each leaf of this foliation.

Let us now make our discussion a little more explicit by describing the subspace $\mathcal{C} \subset T^*\mathcal{Q}$ by a set of M linearly independent phase space functions $\{\psi_m(\mathbf{x})\}$. That is, $\mathcal{C} = \{\mathbf{x} \in T^*\mathcal{Q} : \psi_m(\mathbf{x}) = 0 \ \forall m\}$. These functions are often called the ‘primary constraints’ of the system. Given these constraint functions we can neatly describe the foliation of $T\mathcal{Q}$ that we defined earlier. Suppose two nearby points $(\mathbf{q}, \dot{\mathbf{q}})$ and $(\mathbf{q}, \dot{\mathbf{q}} + \epsilon\mathbf{z})$ are in the same leaf of the foliation — that is to say, $\mathbf{z}$ is a tangent vector to that leaf. Then $\mathbf{P}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{P}(\mathbf{q}, \dot{\mathbf{q}} + \epsilon\mathbf{z})$ and so

$$\frac{\partial P_n}{\partial \dot{q}_m} z_m = 0. \tag{2.6}$$

Hence, the tangent vectors to the foliation of $T\mathcal{Q}$ are just the null eigenvectors of the Hessian matrix W_{mn}. It is easy to describe a basis for these null eigenvectors. Recall that $F^*\psi_m(\mathbf{q}, \dot{\mathbf{q}}) = \psi_m(\mathbf{q}, \mathbf{P}(\mathbf{q}, \dot{\mathbf{q}})) = 0$, such that

$$\frac{\partial F^*\psi_a}{\partial \dot{q}_n} = \frac{\partial \psi_a}{\partial p_m} \frac{\partial P_m}{\partial \dot{q}_n} = 0. \tag{2.7}$$

So, the vectors $\partial\psi_a/\partial p_n$ are null eigenvectors of the Hessian matrix. Moreover, the dimension of $\ker F$ is M, the number of primary constraints, and so the vector fields

$$z_a \equiv \frac{\partial \psi_a}{\partial p_m} \frac{\partial}{\partial \dot{q}_m} \tag{2.8}$$

form a basis for the null space of W_{mn}.

The Dynamics of a Hamiltonian System

Recall that the dynamics generated by the action can also be described, for a theory with an invertible Hessian matrix, by the Hamiltonian equations of motion

$$\dot{A}(\mathbf{x}, t) = XA(\mathbf{x}, t) = \frac{\partial A(\mathbf{x}, t)}{\partial t} + [A(\mathbf{x}), H(\mathbf{x})], \tag{2.9}$$

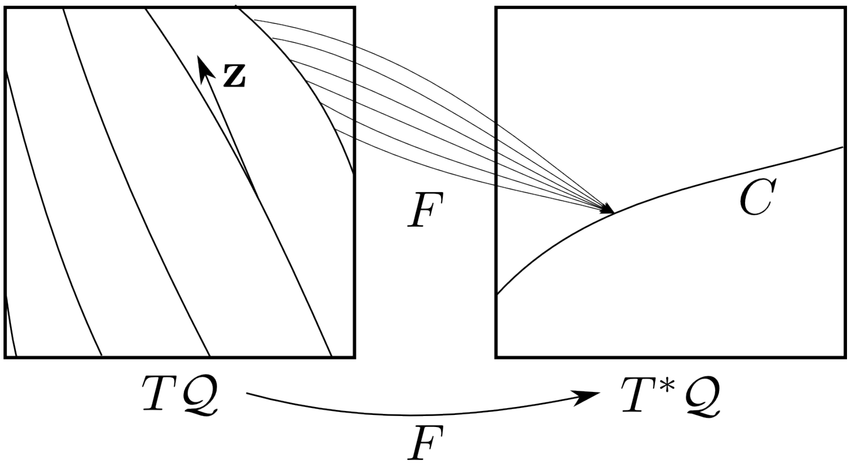

Figure 2.1: The Legendre transformation, F, between the tangent bundle and cotangent bundle. The image of F is a subspace C of $T^*\mathcal{Q}$ and F^{-1} defines a foliation of $T\mathcal{Q}$.

where $X \equiv \partial_t + [*, H]$ is called the *Hamiltonian vector field*, $[*, *]$ is the Poisson bracket on $T^*\mathcal{Q}$ (defined in the usual way) and $H(\mathbf{x})$ is a phase space function that pulls back to the energy function $E(\mathbf{q}, \dot{\mathbf{q}}) = \dot{q}_m P_m(\mathbf{q}, \dot{\mathbf{q}}) - L(\mathbf{q}, \dot{\mathbf{q}})$ under F. The function $H(\mathbf{x})$ is called the *Hamiltonian* and, following our previous discussion, one might worry that the Hamiltonian does not always exist. However, by (2.4)–(2.8)

$$z_a E(\mathbf{q}, \dot{\mathbf{q}}) = \frac{\partial \psi_a}{\partial p_m} P_m(\mathbf{q}, \dot{\mathbf{q}}) + \frac{\partial \psi_a}{\partial p_m} \dot{q}_n W_{mn} - \frac{\partial \psi_a}{\partial p_m} \frac{\partial L}{\partial \dot{q}_m} = 0 \tag{2.10}$$

such that E is constant along each leaf of the $T\mathcal{Q}$ foliation and, consequently, a Hamiltonian satisfying $F^* H(\mathbf{x}) = E(\mathbf{q}, \dot{\mathbf{q}})$ can always be defined. There is, however, some arbitrariness in our definition of $H(\mathbf{x})$ since, for any functions $\lambda_a(t)$, $F^* H = F^*(H + \lambda_a \psi_a)$ so that $H + \lambda_a \psi_a$ is just as good a Hamiltonian as H. But these two Hamiltonians generate different dynamics. It was for this reason that Dirac included the λ_a as variables in his theory and introduced the *primary Hamiltonian*, $H_P = H + \lambda_a(t)\psi_a$. This notation is developed by Henneaux and Teitelboim [19]. We will not adopt this notation. Instead, since dynamical gauge freedom is intuitively an arbitrariness in the time evolution of the theory, we will not modify the Hamiltonian but rather the equations of motion. For a Hamiltonian gauge theory with primary constraints $\{\psi_a\}$ we will define the Hamiltonian vector fields by

$$X = \partial_t + [*, H] + [*, \lambda_a(t)\psi_a], \tag{2.11}$$

for arbitrary, undetermined functions $\lambda_a(t)$. This notation is similar to that adopted in Refs. [22]–[24].

The Consistency Algorithm

It is often the case that (2.11) is not a particularly convenient form in which to express the time evolution of the system. This is because (2.11) must be solved subject to the conditions $\psi_m(\mathbf{x}) = 0$. If we wanted to solve the equations of motion this way, we would have to be careful to choose initial data $\mathbf{x}_0$ and functions $\lambda_a(t)$ such that our solutions $\mathbf{x}(t)$ satisfy $\psi_m(\mathbf{x}(t)) = 0$ for all time t. It would be better to find a new constraint surface $\mathcal{C}_C$ and a new set of Hamiltonian vector fields X_C so that, given some initial data on $\mathcal{C}_C$, the evolution generated by X_C remains on $\mathcal{C}_C$ for all time. I will call this the *consistent description* of the dynamics and there is a simple algorithm, proposed by Dirac, to find both $\mathcal{C}_C$ and X_C.

Let us begin with a set of constraints $\Psi = \{\psi_a\}$ and a Hamiltonian vector field $X = \partial_t + [*, H] + [*, \lambda_a(t)\psi_a]$. A constraint function $\psi \in \Psi$ is called *first class* if $[\psi, \psi_a]$ vanishes on the constraint surface for all a, while ψ is called *second class* if it is not first class. Let $\{\psi_b\}_{b\in\mathbb{B}} \subset \Psi$ be the set of first class constraints and $\{\psi_c\}_{c\in\mathbb{C}} \subset \Psi$ be the set of second class constraints. The first iteration of the algorithm then proceeds in two steps:

> **Step A.** Demanding that time evolution preserves each of the second class constraints yields the conditions
>
> $$X\psi_c = [\psi_c, H] + \lambda_{c'}(t)[\psi_c, \psi_{c'}] \approx 0, \tag{2.12}$$
>
> where c and c' run over the second class constraints and '$\approx$' denotes equality on the constraint surface. We can solve these conditions by putting $\lambda_c(t) = -M^{-1}_{cc'}[\psi_{c'}, H]$, where $M^{-1}_{cc'}$ is the inverse of the matrix $[\psi_c, \psi_{c'}]$. Substituting this into X gives the new Hamiltonian vector fields
>
> $$X_{(1)} = \partial_t + [*, H_{(1)}] + \lambda_b(t)[*, \psi_b], \tag{2.13}$$
>
> where we have defined $H_{(1)} = H - M^{-1}_{cc'}[\psi_{c'}, H]\psi_c$.
>
> **Step B.** Demanding that time evolution preserves the first class constraints gives the conditions
>
> $$X_{(1)}\psi_b = [\psi_b, H_{(1)}] = [\psi_b, H] \approx 0, \tag{2.14}$$
>
> where b runs over the primary constraints. Some of the Poisson brackets might vanish identically. For those brackets that do not vanish, we ensure that (2.14) is satisfied on the constraint surface by introducing additional constraints, $\omega_b = [\psi_b, H]$. We will call the enlarged set of constraints $\Psi_{(1)}$ and the corresponding constraint surface $\mathcal{C}_{(1)}$.

One then iterates this process. The algorithm terminates when we find $\Psi_{(n)}$ and $X_{(n)}$ such that $X_{(n)}\psi = 0$ for all $\psi \in \Psi_{(n)}$. Given this, it is not difficult to show that the constraints appearing together with arbitrary multipliers in $X_{(n)}$ are precisely the primary constraint that are first class with respect to $\Psi_{(n)}$. Moreover, we have the following result.

Proposition 1. *For a Hamiltonian theory on a finite dimensional configuration space, the consistency algorithm terminates after finitely many iterations.*

Proof. Suppose that the configuration space of the theory $\mathcal{Q}$ has dimension N. Then, the initial constraint surface $\mathcal{C}$ described by the constraint set Ψ must have some dimension M with $0 \leq M \leq 2N$. Now, suppose that the consistency algorithm never terminates. An immediate consequence is that, for each iteration of the algorithm, new constraints must be generated. If, in the n^{th} iteration of the algorithm, new constraints are not generated, then the algorithm will terminate on the next iteration because Step A of the algorithm ensures that $X_{(n+1)}\psi \approx 0$ for all $\psi \in \Psi_{(n)} = \Psi_{(n+1)}$. Thus, every iteration of the algorithm generates a new constraint. Moreover, each additional constraint must decrease the dimension of the constraint surface (otherwise the new constraint would vanish on the old constraint surface and Step B would not have generated the new constraint in the first place). Consequently, after $M+1$ steps of the algorithm the dimension of the constraint surface, $\mathcal{C}_{(M+1)}$, is negative. This is a contradiction. $\square$

Many theories do not have a finite dimensional configuration spaces. Nonetheless, for most relevant theories we are still guaranteed that the algorithm will terminate. For instance, in electromagnetism the configuration space comprises all the 4-vector fields $A^\mu(x)$ on Minkowski space. However, for each fixed point x in Minkowski space, we can consider $A^\mu(x)$ as existing in a finite, 4-dimensional configuration space and constrained by a finite dimensional constraint surface. The argument presented above can then be applied to this finite dimensional situation to show that the algorithm terminates "at x". But, for course, this holds for all x and so the algorithm terminates for electromagnetism.

2.2 The Dirac Conjecture Problem

In our discussion thus far we have addressed the dynamical gauge freedom of Hamiltonian systems by studying the geometry of the Legendre transformation and by carefully considering the equations of motion. Now we would like to determine precisely which states are physically equivalent by studying *gauge transformations* — transformations on $T^*\mathcal{Q}$ that do not alter the physical state of the system. The orbits of the gauge transformations define subsets of $T^*\mathcal{Q}$ that correspond to individual physical states.

Dirac proposed that each first class constraint ψ_b generates a gauge transformation via $\delta\mathbf{x} = [\mathbf{x}, \psi_b]$. This is now known as 'the Dirac conjecture' and it is found in textbooks such as Henneaux and Teitelboim [19]. Dirac's argument (as presented in [18] and refined in [19]) begins by observing that the equations of motion are

$$\dot{\mathbf{x}} = [\mathbf{x}(t), H] + [\mathbf{x}(t), \lambda_b(t)\psi_b], \tag{2.15}$$

where b runs over the first class constraints. We pause here to consider the first flaw in the argument. Namely, that these are not, in general, the equations of motion. Dirac modified the Hamiltonian dynamics by including all first class constraints in the equations of motion together with arbitrary functions of time. However, as follows from the previous section's analysis, the equations of motion — in the consistent description — are $\dot{\mathbf{x}} = [\mathbf{x}(t), H] + \lambda_d(t)[\mathbf{x}(t), \psi_d]$, where d runs over the *primary* first class constraints. This notwithstanding, let us attempt to continue Dirac's argument as best we can.

Choose some initial point $\mathbf{x}_0$ that satisfies the constraints. Following Dirac, we can now evolve the system forward by an infinitesimal time ϵ using (2.15) with either $\lambda_b(t)$ or $\lambda'_b(t) = \lambda_b(t) + \delta\lambda_b(t)$ (since the λ_b are arbitrary). Doing this gives us two physically equivalent points in phase space, $\mathbf{x}(\epsilon)$ and $\mathbf{x}'(\epsilon)$, which are, to first order in ϵ, given by

$$\mathbf{x}(\epsilon) = \mathbf{x}_0 + [\mathbf{x}, H]|_{\mathbf{x}_0}\,\epsilon + \lambda_b\,[\mathbf{x}, \psi_b]|_{\mathbf{x}_0}\,\epsilon, \tag{2.16}$$

and

$$\mathbf{x}'(\epsilon) = \mathbf{x}_0 + [\mathbf{x}, H]|_{\mathbf{x}_0}\,\epsilon + \lambda'_b\,[\mathbf{x}, \psi_b]|_{\mathbf{x}_0}\,\epsilon. \tag{2.17}$$

The difference between these two physically equivalent points is $\delta\mathbf{x} = \epsilon\,\delta\lambda_b[\mathbf{x}, \psi_b]$. From this Dirac concluded that the transformation $\mathbf{x} \to \mathbf{x} + \delta\mathbf{x}$ is a gauge transformation and, consequently, that all first class constraints generate gauge transformations. This brings us to the second flaw in the argument. It is not necessarily the case that all nearby, physically equivalent points in phase space must be related by some infinitesimal time evolution from one point in phase space. Two physically equivalent points might evolve from two different points in phase space — see Figure 2.2. Indeed, one need only take the two distinct points $\mathbf{x}(\epsilon)$ and $\mathbf{x}'(\epsilon)$ calculated above. Arbitrary evolutions of these points by a further $\delta t = \epsilon$ gives

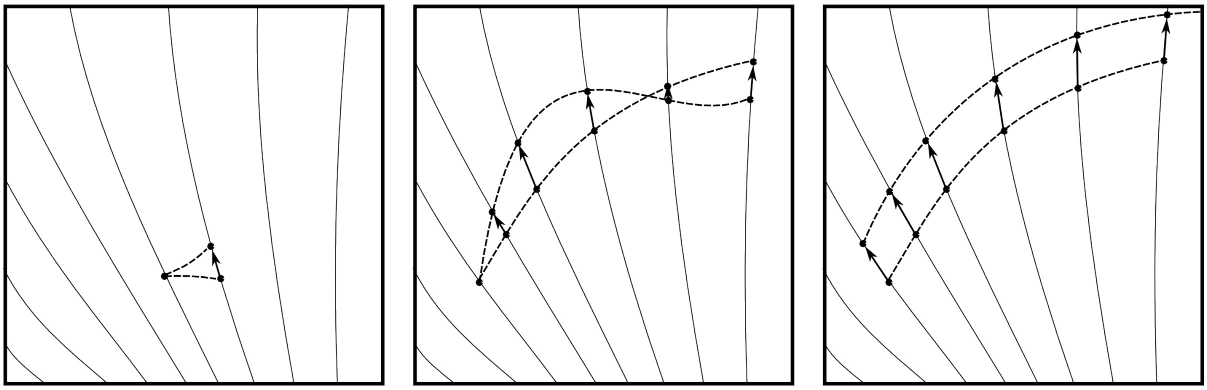

Figure 2.2: Gauge transformations in phase space. *Physically equivalent points lie on gauge orbits (thin lines) generated by gauge transformations (arrows). Dirac considered physically equivalent points arising from the same point in phase space (left pane) following an infinitesimal evolution in time (dashed lines). Pons considered physically equivalent points arising dynamically from an arbitrary time evolution (centre pane). We consider, in addition, rigid symmetries of the evolution (right pane).*

two physically equivalent points not simply related by the first-class generator proposed by Dirac.

2.3 Gauge Freedom without Dirac

Let us now investigate the dynamical gauge freedom of a Hamiltonian system without making the error identified in the previous section. Once again, consider a Hamiltonian system in consistent form with a constraint set Ψ and Hamiltonian vector field

$$X = \partial_t + [*, H] + \lambda_c(t)[*, \psi_c], \tag{2.18}$$

where the index c runs over the first-class primary constraints.

Theorem 1. *Let J be a first-class function. J generates gauge transformations if and only if $\partial_t J + [J, H]$ is a linear combination of the primary first-class constraints (up to the addition of an arbitrary function of time).*

Proof. We proceed by analysing the time-independent and time-dependent cases separately. Suppose that the transformation $\delta\mathbf{x} = \epsilon[\mathbf{x}, J]$ is a time-independent gauge transformation. That is, for any points on the constraint surface $\mathbf{x}(0)$ and $\mathbf{x}'(0) = \mathbf{x}(0) + \epsilon\, [\mathbf{x}, J]|_{\mathbf{x}(0)}$, their subsequent time evolutions, $\mathbf{x}(t)$ and $\mathbf{x}'(t)$, satisfy $\mathbf{x}'(t) = \mathbf{x}(t) + \epsilon\, [\mathbf{x}, J]|_{\mathbf{x}(t)}$ for all t. Since $\mathbf{x}(t)$ and $\mathbf{x}'(t)$ satisfy this condition by construction at $t = 0$, it is equivalent to demand that the time derivatives of both sides are equal. We find,

$$\begin{aligned} \text{LHS} &= [\mathbf{x}, H]|_{\mathbf{x}'(t)} + [\mathbf{x}, \lambda_c\psi_c]|_{\mathbf{x}'(t)}\,, \\ \text{RHS} &= [\mathbf{x}, H]|_{\mathbf{x}(t)} + [\mathbf{x}, \lambda_c\psi_c]|_{\mathbf{x}(t)} + \epsilon\,[[\mathbf{x}, J], H]|_{\mathbf{x}(t)} + \epsilon\,[[\mathbf{x}, J], \lambda_c\psi_c]|_{\mathbf{x}(t)}\,. \end{aligned} \tag{2.19}$$

Setting both sides equal and applying the Jacobi identity gives us the condition that

$$\frac{\partial[\mathbf{x}, H]}{\partial x^m}[x^m, J] + \frac{\partial[\mathbf{x}, \lambda_c\psi_c]}{\partial x^m}[x^m, J] + [[J, H], \mathbf{x}] + [[H, \mathbf{x}], J] + [[\lambda_c\psi_c, \mathbf{x}], J] \tag{2.20}$$

must vanish when evaluated at any $\mathbf{x}(t)$. However, since we can choose any initial data and any functions $\lambda_c(t)$, $\mathbf{x}(t)$ can be any point on the constraint surface. Thus, the condition is simply that (2.20) vanishes on the constraint surface. Moreover, by the definition of the Poisson bracket we have, for any phase space function A,

$$\frac{\partial[\mathbf{x}, A]}{\partial x^m}[x^m, J] = \frac{\partial[\mathbf{x}, A]}{\partial x^m}\Omega^{mn}\frac{\partial J}{\partial x^n} = [[\mathbf{x}, A], J], \tag{2.21}$$

where Ω^{mn} is the symplectic matrix. This allows us to re-express (2.20) as

$$[[J, H], \mathbf{x}]. \tag{2.22}$$

However, $[[J, H], \mathbf{x}] \approx 0$ iff $\partial_{x^m}[J, H] = 0$ for all x^m, i.e., iff $[J, H] = 0$, up to the addition of a function of time. Thus we conclude: *J generates a non-dynamical gauge freedom iff $[J, H] = 0$ up to an arbitrary function of time.*

Now let us consider dynamical gauge freedom. Suppose K generates dynamical gauge transformations. Explicitly, this means that: there exist some physically equivalent points $\mathbf{y}(0)$ and $\mathbf{y}'(0)$ such that the time evolution of these points, $\mathbf{y}(t)$ and $\mathbf{y}'(t)$ (obtained using similar functions $\lambda_c(t)$ and $\lambda'_c(t)$), are related by $\mathbf{y}'(t) = \mathbf{y}(t) + \epsilon\, [\mathbf{x}, K]|_{\mathbf{y}(t)}$. We will write $\lambda'_c(t) = \lambda_c(t) + \epsilon\gamma_c(t)$, for small ϵ and arbitrary $\gamma_c(t)$, and their associated Hamiltonian vector fields will be denoted X and X', respectively. Now, K satisfies this condition if the time evolution of $\mathbf{y}'(t)$ under X' is equal to the time evolution of $\mathbf{y}(t) + \epsilon\, [\mathbf{x}, K]|_{\mathbf{y}(t)}$ under X for all t. So, evaluating each term we find

$$\begin{aligned}
\text{LHS} &= X'\mathbf{y}'(t) \\
&= [\mathbf{x}, H]|_{\mathbf{y}'(t)} + [\mathbf{x}, \lambda_c\psi_c]|_{\mathbf{y}'(t)} + \epsilon\, [\mathbf{x}, \gamma_c\psi_c]|_{\mathbf{y}'(t)}\,, \\
\text{RHS} &= X\left(\mathbf{y}(t) + \epsilon\, [\mathbf{x}, K]|_{\mathbf{y}(t)}\right) \\
&= [\mathbf{x}, H]|_{\mathbf{y}(t)} + [\mathbf{x}, \lambda_c\psi_c]|_{\mathbf{y}(t)} + \epsilon\, [\mathbf{x}, \partial_t K]|_{\mathbf{y}(t)} + \epsilon\, [[\mathbf{x}, K], H]|_{\mathbf{y}(t)} \\
&\qquad\qquad\qquad\qquad + \epsilon\, [[\mathbf{x}, K], \lambda_c\psi_c]|_{\mathbf{y}(t)}\,.
\end{aligned} \tag{2.23}$$

Setting both sides equal gives the condition that

$$\begin{aligned}
\epsilon\frac{\partial[\mathbf{x}, H]}{\partial x^m}[x^m, K] + \epsilon\frac{\partial[\mathbf{x}, \lambda_c\psi_c]}{\partial x^m}[x^m, K] + \epsilon[\mathbf{x}, \gamma_c\psi_c] + \mathcal{O}(\epsilon^2) \\
= \epsilon[\mathbf{x}, \partial_t K] + \epsilon[[\mathbf{x}, K], H] + \epsilon[[\mathbf{x}, K], \lambda_c\psi_c],
\end{aligned} \tag{2.24}$$

when evaluated at any $\mathbf{y}(t)$ on the constraint surface. As with the non-dynamical case, we now apply (2.21), the Jacobi identity and the first-class property of K. We obtain the equivalent condition that

$$[\partial_t K + [K, H] - \gamma_c\psi_c, \mathbf{x}] \tag{2.25}$$

vanishes on the constraint surface. That is, up to the addition of a function of time,

$$\partial_t K + [K, H] = \gamma_c\psi_c. \tag{2.26}$$

This is what was to be demonstrated. Moreover, note that the condition on non-dynamical generators, $[J, H] = 0$, is a special case of this condition, so the theorem holds for both cases. □

Corollary 1. *A theory with no primary first-class constraints (when in consistent form) has* *no dynamical gauge freedoms.*

Proof. Let $\{\psi_a\}$ be the set of first-class constraints and suppose that $J = \alpha^a(t)\psi_a$ is a gauge generator. Then, by the Theorem, $0 = \partial_t J + [J, H] = \dot{\alpha}^a(t)\psi_a + \alpha^a(t)\partial_t\psi_a + \alpha^a(t)[\psi_a, H]$. But $X\psi^a = \partial_t\psi^a + [\psi^a, H] = 0$ since the theory is in consistent form. So, $\dot{\alpha}^a = 0$ and the gauge generator J is not time-dependent. □

A note on Henneaux and Teitelboim

In chapter one of their book, Henneaux and Teitelboim [19] attempt to prove that every primary first-class constraint is a gauge generator using Dirac's argument. They then conjecture "[...] in general, that all first-class constraints generate gauge transformations." In chapter three they offer a curious "proof" of this conjecture given certain assumptions. These assumptions are such that, under their version of the consistency algorithm, the n^{th} generation of first-class constraints, $\{\phi_{a_n}\}$, can all be written in terms of the Poisson brackets $[\phi_{a_{n-1}}, H]$, where $\{\phi_{a_{n-1}}\}$ is the $(n-1)^{\text{th}}$ generation of constraints. This way, if one assumes that the primary first-class constraints $\{\phi_{a_1}\}$ are gauge generators, then (they claim) all further first-class constraints inherit this property as well. Of course, it is not the case that all primary first-class constraints are gauge generators and consequently this proof of the Dirac conjecture cannot hold. However, there is a second error in the proof: it assumes the result that if ϕ is a gauge generator then so is $[\phi, H]$. This claim is not true (electromagnetism is a counter example, as discussed in Appendix A). Their proof of it in chapter one follows from an analysis of infinitesimal time-evolutions and so fails for the same reason as discussed in Section 2.2. Nonetheless, we can prove the following weakened version of their claim which is realised, for instance, in Hamiltonian GR.

Proposition 2. *Let H be a time-independent Hamiltonian that can be written as a linear combination of the constraints. Then, for every gauge generator J, $[J, H]$ is also a gauge generator. Alternatively, this also holds if H is a (possibly time-dependent) linear combination of the primary first-class constraints.*

Proof. For the first case observe that $\partial_t[J, H] + [[J, H], H] = [\partial_t J + [J, H], H] = 0$. The second case is similarly immediate. □

2.4 Gauge Fixing

In a gauge theory each physical state is represented by a subset of $T^*\mathcal{Q}$ defined by the orbits of the gauge transformations. But suppose we can find a surface $\mathcal{C}_{(GF)}$ in $\mathcal{C}$ that intersects each of the gauge orbits precisely once. This surface, by construction, can be identified with the set of physical states and we can remove all gauge freedom from the theory by constraining it to $\mathcal{C}_{(GF)}$. The process is called gauge fixing.

To be more explicit, consider a theory, in consistent form, that has a constraint set Ψ and Hamiltonian vector field $X = [*, H] + \lambda_d(t)[*, \psi_d]$ for some $H(x)$. Suppose further that the gauge freedoms of this theory can be described by some set of generators $\{G_i\}$. We begin by removing dynamical gauge freedom from the equations of motion. Recalling Step A of the consistency algorithm, we can fix the functions $\lambda_d(t)$ by introducing some additional constraints $\{\omega_c\}$ such that the matrix $M_{cd} = [\omega_c, \psi_d]$ is square and invertible. Now, the generators $\{G_i\}$ are made consistent with these new constraints by demanding that the brackets $\delta\omega_c = [\omega_c, G_i]$ all vanish on the constraint surface — this will remove some of the generators from our theory. However, some gauge generators $\{G_j\} \subset \{G_i\}$ may remain.

Since the equations of motion for the theory are now determinate, the $\{G_j\}$ must generate non-dynamical gauge transformations. This non-dynamical freedom can be removed from the theory by introducing further constraints. As an example of this process, see Appendix A for a complete gauge fixing of electromagnetism.

2.5 Equivalent Gauge Theories

In the preceding sections we developed a formalism for describing theories with gauge freedom. However, since gauge freedom is non-physical, it is possible for two theories with different gauge freedoms to describe the same physical dynamics. We will call such theories equivalent, and in this subsection we will formalise this idea. In Chapter 4 we will use the results from this section to construct shape dynamics.

We will say that two gauge theories T and T' are *equivalent* if they can both be gauge fixed to the same theory. This gauge fixed theory, common to both T and T', is called the *dictionary theory*. It is not difficult to construct two equivalent gauge theories. Suppose we have some gauge theory T_L together with two partial gauge fixings of this theory, GF1 and GF2. Define T_1 as the theory obtained by applying $GF2$ to T_L and define T_2 as the theory obtained by applying $GF1$. There is nothing in the gauge fixing procedure that depends on the order in which gauge fixing constraints are applied. Consequently, T_1 and T_2 are equivalent because gauge fixing T_1 with GF1 and T_2 with GF2 both give rise to the same theory — call it T_{dict}. The theory T_L is called a *linking theory* by Koslowski and Gomes, and given such a theory it is easy to construct equivalent theories in this way.

The Linking Theory Construction

Now let us consider the inverse problem. Suppose we have a theory T_1 and we would like to construct an equivalent theory T_2 that has different gauge freedoms. To accomplish this we can follow Koslowski and Gomes [25] in constructing a *special linking theory* from T_1. Their presentation of this construction makes explicit use of the Dirac formalism, but their idea can be expressed in more general terms using the formalism we are using here. Let T_1 be a gauge theory on a phase space $\mathcal{P}$. We can construct a linking theory, T_L, from T_1 in the following way. Define the extended phase space by the direct sum $\mathcal{P}_L = \mathcal{P} \oplus \tilde{\mathcal{P}}$ where $\tilde{\mathcal{P}}$ is a two-dimensional symplectic space that we can describe using coordinates (ϕ^m, π_m). The Poisson bracket is defined on $\mathcal{P}_L$ by requiring that $[\phi^m, \pi_n] = \delta^m_n$ and extending this to the whole space in the usual way. Finally, we extend T_1 to a theory on $\mathcal{P}_L$ which we will call T_L. We want this new theory to be fully equivalent to T_1. Motivated by this we define T_L as T_1 together with new constraints $B_m = \pi_m$. If T_1 has Hamiltonian vector field X then the dynamics of T_L is generated by

$$X_L = X + [*, \alpha^m \pi_m], \tag{2.27}$$

for some arbitrary functions $\alpha^m(t)$. The gauge fixing constraints $\phi^m = 0$ reduce T_L to a theory, call it T_{L1}, that matches T_1 on the surface $\phi^m = 0, \pi_m = 0$ in $\mathcal{P}_L$. But this surface is trivially isomorphic to $\mathcal{P}$ and so we can identify T_{L1}, a theory on the extended phase space, with T_1. This identification is called *phase space reduction*.

Thus far we have trivially extended T_1 onto an extended phase space $\mathcal{P}_L$. This is not of much use on its own. However, in some cases we can use the extended theory T_L to introduce a new gauge freedom. Suppose we want the one-parameter group $G : \mathbf{q} \mapsto \mathbf{F}(\mathbf{q}, \tau)$ to be a gauge freedom of T_L. Suppose in particular that $\tau = 0$ corresponds to the identity map.

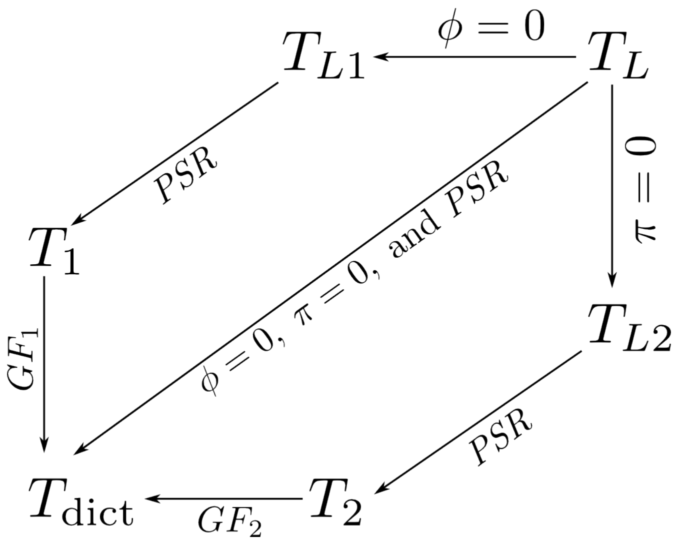

Figure 2.3: The special linking theory construction. *The linking theory T_L is obtained by extending T_1 onto an extended phase space which includes new variables (ϕ^i, π_i). T_1 can be obtained from T_L by gauge fixing with $\phi = 0$ and performing phase space reduction (PSR). A second theory, T_2, can be obtained by gauge fixing with $\pi = 0$ and performing PSR. By construction, T_1 and T_2 are equivalent theories and can be further gauge fixed to some common dictionary theory T_{dict}.*

Now, since ϕ is arbitrary in T_L the coordinates $\mathbf{q}' = \mathbf{F}(\mathbf{q}, \phi)$ exhibit the gauge freedom G. Further, if we transform to these G-invariant coordinates using a canonical transformation on $\mathcal{P}_L$ the resulting theory will be equivalent to T_1. In particular, $\mathbf{q}' = \mathbf{q}$ on the surface $\phi = 0$ — this ensures that T_L can still be gauge fixed to T_1. The non-triviality of this construction emerges because the canonical transformation modifies the constraints $B_m = 0$. This means that gauge fixing with $\pi_m = 0$ might give a new gauge theory, call it T_2. Indeed, if $\pi_m = 0$ does not gauge fix the G-invariance of T_L, then T_2 will have the desired G gauge freedom. This construction is neatly summarised by the diagram in Figure 2.3. Of course, this approach does not always yield non-trivial results (otherwise, all gauge theories on the same phase space would be equivalent).

In reference [25], Koslowski and Gomes attempt an analysis of the possible outcomes of this construction using the language of Dirac constraints. For instance, they consider the Hamiltonian description of a particle moving in a 3-dimensional, spherically symmetric potential. If we try to introduce z-axis rotations as a dynamical gauge freedom, we find that the linking construction fails. This calculation is performed in Appendix B. Koslowski and Gomes attribute this failing to the observation that the $B = \pi$ constraint is first class. However, it might be more physically interesting to note that the original theory has no dynamical gauge-freedom to begin with. Indeed, all of the examples in ref. [25] involve theories that do not have dynamical gauge-freedom — quite unlike electromagnetism and GR. It seems, then, that more analysis of the outcomes of the linking formalism is needed.

Chapter 3

Gravity as a Gauge System

General relativity is a theory of a 4-dimensional Lorentzian manifold. In this description we can introduce the notion of time by choosing a particular reference frame, i.e. by choosing, at each point, a particular time-like vector which defines a temporal direction. However, introducing time locally is not sufficient if we wish to implement a Hamiltonian formulation of the theory. For this we need a global time parameter. Such a global time can be introduced by choosing some function $t : \mathcal{M} \to \mathbb{R}$ such that if p is causally prior to q, in some sense, then $t(p) < t(q)$. Such a function is called a time function and it defines a foliation of $\mathcal{M}$ into spatial hypersurfaces of constant t. Given such a foliation, we can redefine the 4-metric and 4-curvature in terms of 3-dimensional quantities that vary with t. This is called a '3+1'-split and this formalism is sketched in Section 3.1. We then apply this to deriving the Hamiltonian formulation of GR in Section 3.2. Finally, in Section 3.3 we conduct some original analysis of its gauge freedoms using the tools developed in the previous chapter.

3.1 Breaking Spacetime

For a globally hyperbolic Lorentzian manifold $\mathcal{M}$, with metric $\mathbf{g}$, we can find a time function $t : \mathcal{M} \to I \subset \mathbb{R}$, for some interval I, so that each $t =$ const. defines a spatial hypersurface in $\mathcal{M}$. Call these hypersurfaces Σ_t and suppose that these comprise a foliation of $\mathcal{M}$. Now, the pullback of the metric $\mathbf{g}$ onto any of these hypersurfaces defines an induced metric, $\mathbf{h}$, on the hypersurfaces. In components, the metric $\mathbf{g}$ has four more free entries than $\mathbf{h}$. This missing information can be specified by defining a scalar field N and vector field $\mathbf{N}$ on the hypersurfaces. Explicitly, following Arnowitt, Deser and Misner [3] we take $N = (-g^{00})^{(-1/2)}$ and $N_i = g_{0i}$. These quantities are called the lapse and the shift, respectively.

At each point $p \in \mathcal{M}$, the foliation Σ_t defines a time-like unit vector $\mathbf{u} \in T_p\mathcal{M}$ that is normal to the appropriate hypersurface. We can use $\mathbf{u}$ to define a projection operator, $\tilde{\mathbf{h}}$. In coordinates, $\tilde{h}^{\nu}_{\mu} = \delta^{\nu}{}_{\mu} + u_\mu u^\nu$. This operator projects all vectors in $T_p\mathcal{M}$ onto the subspace of $T_p\mathcal{M}$ that is tangent to the appropriate hypersurface. Of course, we can easily extend the action of $\tilde{\mathbf{h}}$ to tensors by demanding that $\tilde{\mathbf{h}}(T_1 \otimes T_2) = \tilde{\mathbf{h}}T_1 \otimes \tilde{\mathbf{h}}T_2$. In particular, we are interested in how $\mathbf{u}$ varies on the hypersurfaces, because this contains curvature information. So, we study the shape operator, which in coordinates is $\nabla_\mu u_\nu$, and its spatial projection, $K_{\mu\nu} = \tilde{h}^{\lambda}_{\mu}\tilde{h}^{\rho}_{\nu}\nabla_\lambda u_\rho$. Here ∇ is the Levi-Civita connection associated to $\mathbf{g}$. The tensor $K_{\mu\nu}$ is called the extrinsic curvature or second fundamental form. When pull-backed to the spatial hypersurfaces, the extrinsic curvature can be expressed in terms of spatial quantities as

$$K_{ab} = \frac{1}{2}\pounds_u h_{ab}, \quad \text{or} \quad K_{ab} = \frac{1}{2N}\left(\dot{h}_{ab} - 2\ {}^3\nabla_{(a}N_{b)}\right), \tag{3.1}$$

where ${}^3\nabla$ is the connection defined by $\mathbf{h}$. Finally, the induced metric $\mathbf{h}$ also defines a Ricci scalar ${}^3R$. This is related to the Ricci scalar corresponding to $\mathbf{g}$, ${}^4R$, by the contracted Gauss equation:

$$ {}^4R = {}^3R + K_{ab}K^{ab} - K^2, \tag{3.2} $$

which will be particularly useful for the next section.

3.2 Hamiltonian General Relativity

In a vacuum, general relativity is described by extremising the Einstein-Hilbert action

$$ S[g] = \int \mathrm{d}^4x \, \sqrt{-g} \, {}^4R. \tag{3.3} $$

In order to get a Hamiltonian description of this theory we need to introduce a time coordinate. Assuming the existence of a time function, we obtain a foliation into spatial hypersurfaces and by (3.1) and (3.2) we can write the Einstein-Hilbert action as

$$ S[h_{ab}, N, N^a] = \int \mathrm{d}t \, \mathrm{d}^3x \, N\sqrt{h} \left[{}^3R + \frac{1}{2N}\left(\dot{h}_{\mu\nu} - 2\, {}^3\nabla_{(\mu}N_{\nu)}\right) K^{\mu\nu} - \left(h^{\mu\nu}K_{\mu\nu}\right)^2 \right]. \tag{3.4} $$

In this form the action permits the definition of a Lagrangian, L, and we can find the conjugate momentum functions by taking variations of this Lagrangian. We find

$$ p^{ab} = \frac{\delta L}{\delta \dot{h}_{ab}} = \sqrt{h}\left(K^{ab} - h^{ab}K\right), \quad M = \frac{\delta L}{\delta \dot{n}} = 0, \qquad M_a = \frac{\delta L}{\delta \dot{n}^a} = 0. \tag{3.5} $$

So, the Legendre transformation generates four primary constraints at each point, $M(x)$ and $M_a(x)$. Moreover, the energy function is

$$ \begin{aligned} E &= \int \mathrm{d}^3x \left(p^{ab}(h,\dot{h})\dot{h}_{ab} + M(h,\dot{h})\dot{N} + M_a(h,\dot{h})\dot{N}^a - \mathcal{L}\right) \\ &= \int \mathrm{d}^3x \left(N\sqrt{h}K_{ab}K^{ab} - N\sqrt{h}K^2 + 2p^{ab}\, {}^3\nabla_{(a}N_{b)} - N\sqrt{h}\, {}^3R\right), \end{aligned} \tag{3.6} $$

where we have used (3.1). We observe that $p_{ab}p^{ab} = h(K_{ab}K^{ab} + K^2)$ and $p^2 = 4hK^2$, so one possible choice of Hamiltonian is

$$ H = \int \mathrm{d}^3x \, \sqrt{h} \left[-N\, {}^3R + \frac{N}{h}\left(p_{ab}p^{ab} - \frac{1}{2}p^2\right) - N_a\, {}^3\nabla_b\left(\frac{1}{\sqrt{h}}p^{ab}\right)\right], \tag{3.7} $$

where we obtained the last term by integrating by parts. We now have a Hamiltonian system on a phase space that we coordinatize by $(h_{ab}, p^{ab}; N, M; N^a, M_a)$. The dynamics of this system is generated by

$$ X = \partial_t + [*, H] + [*, \langle \alpha, M\rangle] + [*, \langle \beta^a, M_a\rangle], \tag{3.8} $$

for some arbitrary functions $\alpha(x,t)$ and $\beta^a(x,t)$. We pause here to note the transition to the continuum limit. We are now considering our variables and constraints as labelled by the points of a 3-manifold, Σ. So, we have passed to the continuum limit by defining $\langle f, g\rangle = \int \mathrm{d}^3x\,(f(x)g(x))$ in place of the discrete sum considered earlier. Moreover, since these are functionals on our canonical variables, the Poisson bracket becomes

$$[F,G] = \int \mathrm{d}^3x \left(\frac{\delta F}{\delta h_{ab}(x)} \frac{\delta G}{\delta p^{ab}(x)} - \frac{\delta G}{\delta h_{ab}(x)} \frac{\delta F}{\delta p^{ab}(x)} + \frac{\delta F}{\delta N(x)} \frac{\delta G}{\delta M(x)} - \text{etc.} \right), \tag{3.9}$$

where F and G are functionals of $h_{ab}(x), p^{ab}(x), N(x), M(x), N^c(x)$, and $M_c(x)$.

Now, the consistency algorithm generates four additional constraints at each point:

$$XM = 0 \quad \Rightarrow \quad S(x) = \sqrt{h}\ {}^3R - \frac{1}{\sqrt{h}} \left(p_{ab}p^{ab} - \frac{1}{2}p^2 \right) = 0, \tag{3.10}$$

and

$$XM_a = 0 \quad \Rightarrow \quad D^a(x) = \sqrt{h}\ {}^3\nabla_b \left(\frac{1}{\sqrt{h}} p^{ab} \right). \tag{3.11}$$

These functions, $S(x)$ and $D^a(x)$, are called the *Hamiltonian* and *momentum* constraints, respectively. In terms of these functions we can express the Hamiltonian as $H = \langle N, S\rangle + \langle N_a, D^a\rangle$.

3.3 Gauge Freedom, the Lapse and the Shift

Consider first-class functionals of the canonical variables, J, of the form

$$J = \langle \gamma, M\rangle + \langle \eta^a, M_a\rangle + \langle \lambda, S\rangle + \langle \theta^a, D_a\rangle, \tag{3.12}$$

where $\gamma(x,t)$, $\eta^a(x,t)$, $\lambda(x,t)$ and $\theta^a(x,t)$ are arbitrary time-dependent functions. It follows immediately that

$$\partial_t J + [J,H] = \langle \dot\gamma, M\rangle + \langle \dot\eta^a, M_a\rangle + \left\langle \dot\lambda, S\right\rangle + \left\langle \dot\theta^a, D_a\right\rangle - \langle \gamma, S\rangle - \langle \eta^a, D_a\rangle. \tag{3.13}$$

By theorem 1, J is a gauge generator iff this expression is some combination of the primary first-class constraints, $M(x)$ and $M_a(x)$. It follows that we have two classes of gauge generators,

$$J_1 = \left\langle \dot\theta^a, M_a\right\rangle + \langle \theta^a, D_a\rangle, \quad \text{and} \quad J_2 = \langle \dot\gamma, M\rangle + \langle \gamma, S\rangle. \tag{3.14}$$

Let us investigate each class of transformations in turn. J_1 generates the transformation

$$\delta h_{ab} = \frac{\delta \langle \theta^c, D_c\rangle}{\delta p_{ab}} = \pounds_\theta h_{ab}, \quad \delta N = 0, \quad \delta N^a = \frac{\delta \left\langle \dot\theta^c, M_c\right\rangle}{\delta M_a} = \dot\theta^a. \tag{3.15}$$

The reader will recognise $\delta h_{ab} = \pounds_\theta h_{ab}$ as the generator of diffeomorphisms of the spatial metric. J_1 also generates a corresponding change of the lapse vector N^a. This change is neglected by most treatments of Hamiltonian GR which assume (in the manner of Dirac-Bergmann) that D_a, being a first-class constraint, is a gauge generator. However, in our analysis we have found that D_a alone is not a dynamical gauge generator. Rather, D_a is only a generator of time-independent gauge transformations on the initial data. Instead, we have found that an appropriate combination of D_a and M_a, namely J_1, is a dynamical gauge generator. This result is not surprising if we recall the geometry of the '3+1'-split. Figure 3.1 shows that for a time-dependent diffeomorphism generated by $\theta^a(t)$, the shift vector must change by an amount equal to the rate of change of $\theta^a(t)$ — which is what we

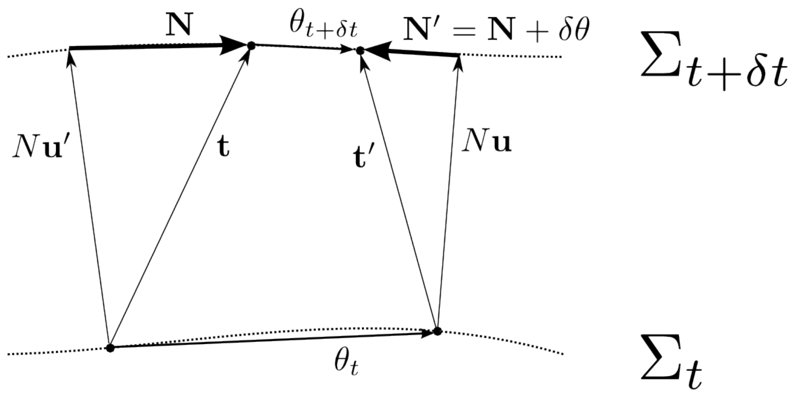

Figure 3.1: Spatial diffeomorphisms in '3+1'. *Spatial diffeomorphisms generated by a time-dependent vector field* $\boldsymbol{\theta}$ *must be accompanied by a change in the lapse vector* $\mathbf{N}$ *in order to correspond to a 4-dimensional diffoemorphism.*

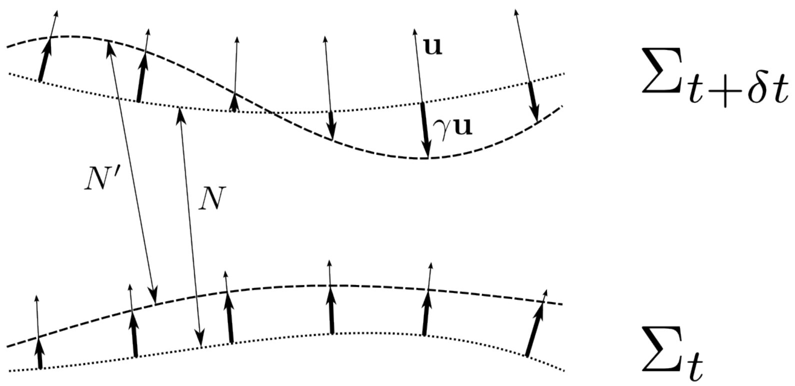

Figure 3.2: Refoliations of the spatial hypersurfaces and corresponding changes to the lapse function N.

found in (3.15). I wish to emphasize here that throwing away that lapse does not, in general, give a theory that is equivalent to GR. Recall that the metric $g_{\mu\nu}$ depends on both h_{ab} and N^a (and N). So, unless N^a is transformed correctly, the transformation $\delta h_{ab} = \pounds_\theta h_{ab}$ does not correspond to a diffeomorphism of the original theory.

J_2 generates the change

$$\delta h_{ab} = \frac{\delta \langle \gamma, S \rangle}{\delta p^{ab}} = \frac{2\gamma}{\sqrt{h}} \left(\frac{1}{2} h_{ab} p - p_{ab} \right), \quad \delta N = \frac{\delta \langle \dot{\gamma}, M \rangle}{\delta M} = \dot{\gamma}, \quad \delta N^a = 0. \tag{3.16}$$

Now, to understand what this means physically, recall the definition of the Legendre transformation. In particular, using (3.5) we can, on the constraint surface, write p_{ab} in terms of the extrinsic curvature so that the transformation (3.16) is $\delta h_{ab} = -2\gamma K_{ab}$. But K_{ab} is the projection of $\pounds_{\mathbf{u}} h_{ab}$ onto the spatial hypersurfaces. Thus, (3.16) is the infinitesimal generator of refoliations. As with our analysis of J_1 we see that the change in the lapse induced by J_2 is physically necessary. It is not enough to drag h_{ab} along the unit normal vectors $\mathbf{u}$: one must also change the lapse function to take into account the 'squishing and pulling' in between the hypersurfaces. If one does not do this then the transformation does not amount to a refoliation, but a change in $g_{\mu\nu}$.

Chapter 4

Shape Dynamics

Shape dynamics is an equivalent theory to Hamiltonian general relativity in the sense that we developed in Section 2.5. That is, both shape dynamics and Hamiltonian GR can be gauge fixed to the same theory and, consequently, both theories should make the same physical predictions. We will construct shape dynamics from Hamiltonion GR on a closed manifold without boundary by using the linking theory construction. Shape dynamics is the theory one obtains from this construction by introducing gauge invariance under volume-preserving conformal transformations.

Shape dynamics was first constructed for a closed, no-boundary manifold by Gomes, Gryb and Koslowski [17] using a symmetry-trading procedure that is similar to the linking theory construction. The linking theory approach was subsequently proposed by Gomes and Koslowski [25] and used to construct shape dynamics for the asymptotically flat case. Our calculation obtains the results in [17] using the linking theory construction; in so doing we "fill the gap" between refs. [17] and [25]. Moreover, in our calculation we do not use the Dirac approach to gauge systems; rather, we make use of the machinery developed in Chapter 2. As such, our results agree in general terms with ref. [25] but there are important differences in how we conduct gauge fixing and express dynamical gauge freedom[1]. This yields some clarifications. For instance, our calculation makes explicit the gauge freedoms of shape dynamics.

4.1 The Linking Theory

Recall from Chapter 3 that Hamiltonian GR is a theory on a phase space $\mathcal{P}$ that we coordinatize by the canonically conjugate pairs (h_{ab}, p^{ab}), (N, M) and (N^a, M_a). At each point in the 3-manifold Σ, the theory has four primary first class constraints: M and M_a. There are also four secondary constraints at each point: $S(x)$ and $D^a(x)$. The reader is referred to (3.10) and (3.11) for the explicit form of these functions. The Hamiltonian is $H = \langle N, S\rangle + \langle N_a, D^a\rangle$, and dynamics is generated by

$$X = \partial_t + [*, H] + [*, \langle \alpha, M\rangle] + [*, \langle \beta_a, M^a\rangle]\,, \tag{4.1}$$

where $\alpha(x,t)$ and $\beta_a(x,t)$ are arbitrary functions. We now construct a linking theory by extending the phase space to include a new degree of freedom at each point, $\phi(x)$, and its conjugate momentum density, $\pi(x)$. Moreover, we add a fifth primary first-class constraint to our theory: $B = \pi(x)$. The Hamiltonian of our linking theory is the same as before.

[1]There are also less important differences: for example, we do not follow Dirac and include constraints in the Hamiltonian on an ad hoc basis.

However, the Poisson bracket is now extended such that $[\phi(x), \pi(y)] = \delta(x-y)$ and the dynamics of our extended theory is generated by

$$X = \partial_t + [*, H] + [*, \langle \alpha, M \rangle] + [*, \langle \beta_a, M^a \rangle] + [*, \langle \gamma, B \rangle]\,, \tag{4.2}$$

for arbitrary scalar field $\gamma(x,t)$.

In order to obtain shape dynamics we now introduce a new gauge symmetry into our linking theory: invariance under volume-preserving conformal transformations (VPCTs). The total volume of $\mathcal{S}$ is $V \equiv \int \mathrm{d}^3x\sqrt{h}$ and we can define the *average* of a scalar field $\alpha(x)$ by

$$\langle \alpha \rangle = \frac{1}{V}\int \mathrm{d}^3x\sqrt{h}\,\alpha(x). \tag{4.3}$$

The conformal transformations that leave V unchanged are all of the form $h_{ab} \mapsto h'_{ab} = \mathrm{e}^{4\hat{\phi}}h_{ab}$ where

$$\hat{\phi} = \phi - \frac{1}{6}\log\left\langle e^{6\phi}\right\rangle, \tag{4.4}$$

and ϕ is any scalar field (see Appendix C for details). To introduce invariance under VPCTs we perform the transformation $h_{ab} \mapsto h'_{ab} = \mathrm{e}^{4\hat{\phi}}h_{ab}$ as part of a canonical transformation on extended phase space generated by the functional

$$J = \int \mathrm{d}^3x\left(\mathrm{e}^{4\hat{\phi}}h_{ab}p'^{ab} + \phi\pi' + NM' + N_aM'^a\right). \tag{4.5}$$

The corresponding transformation equations are

$$h'_{ab} = \frac{\delta J}{\delta p'^{ab}},\quad p^{ab} = \frac{\delta J}{\delta h_{ab}},\quad \phi' = \frac{\delta J}{\delta \pi'},\quad \pi = \frac{\delta J}{\delta \phi}, \tag{4.6}$$

and so on. Solving these equations gives the following expressions for the transformed coordinates:

$$\begin{array}{ll} h'_{ab} = \mathrm{e}^{4\hat{\phi}}h_{ab}, & p'^{ab} = \mathrm{e}^{-4\hat{\phi}}\left[p^{ab} - \frac{1}{3}\left(1-\mathrm{e}^{6\hat{\phi}}\right)\langle p\rangle\, h^{ab}\sqrt{h}\right], \\ \phi' = \phi, & \pi' = \pi + 4\langle p\rangle\sqrt{h} - 4p, \\ N' = N, & M' = M, \\ N'^a = N^a, & M'_a = M_a. \end{array} \tag{4.7}$$

Given this, the transformed constraint functions are

$$\begin{aligned} B' &= \pi + 4\langle p\rangle\sqrt{h} - 4p, \\ S' &= \mathrm{e}^{2\hat{\phi}}\sqrt{h}\,(\,{}^3R - 8\partial_a\partial^a\phi - 8\partial_a\phi\partial^a\phi) \\ &\quad + \mathrm{e}^{-6\hat{\phi}}\tfrac{1}{\sqrt{h}}\left[p_{ab}p^{ab} - \tfrac{1}{2}p^2 + \tfrac{1}{3}\left(1-\mathrm{e}^{6\hat{\phi}}\right)\langle p\rangle\, p\sqrt{h} - \tfrac{1}{6}\left(1-\mathrm{e}^{6\hat{\phi}}\right)^2\langle p\rangle^2 h\right], \\ D'^a &= \mathrm{e}^{6\hat{\phi}}\,{}^3\nabla'_b\left[\mathrm{e}^{-10\hat{\phi}}\sqrt{h}\left(p^{ab} - \tfrac{1}{3}\left(1-\mathrm{e}^{6\hat{\phi}}\right)\langle p\rangle\, h^{ab}\sqrt{h}\right)\right]. \end{aligned} \tag{4.8}$$

The Hamiltonian of the transformed linking theory is $H' = \langle N, S'\rangle + \langle N_a, D'^a\rangle$.

To summarise, we now have a theory on the extended phase space with five primary constraints at each point: $M(x)$, $M_a(x)$, and $B'(x) = \pi + 4\langle p\rangle\sqrt{h} - 4p$. We also have four additional constraints at each point: $S'(x)$ and $D'^a(x)$. The Hamiltonian is $H' =$

$\langle N, S' \rangle + \langle N_a, D'^a \rangle$ and the dynamics is generated by

$$X = \partial_t + [*, H'] + [*, \langle \alpha, M \rangle] + [*, \langle \beta_a, M^a \rangle] + [*, \langle \gamma, B' \rangle] \,. \tag{4.9}$$

Finally, since we have performed a canonical transformation the equations of motion for the transformed variables must take the same form as the original equations of motion. That is, for any functionals on the extended phase space F we must have $X'F' = (XF)'$ where F' denotes the transformed functional (resp. vector field) under our canonical transformation. Since our original theory was in consistent form, it follows that the transformed linking theory is also in consistent form.

Gauge Fixing for Hamiltonian GR

At this point it is instructive to verify that our linking theory can be gauge fixed to Hamiltonian GR. Let us compare Equations (4.1) and (4.9). Our linking theory has one extra dynamical gauge freedom generated by B'. In order to recover Hamiltonian GR we need to remove this additional gauge freedom and recover the constraint surface (which was defined by m, m_a, S and D^a). Of course, by construction we can achieve this by introducing the gauge-fixing constraint $\phi(x) = 0$ at each point. To begin, we make the dynamics consistent with this new constraint by demanding that $X \langle x, \phi \rangle = 0$ which immediately implies that $\tau(t) = 0$. Moreover, observe that the surface defined by a transformed constraint function $F' = 0$ and $\phi = 0$ is identical to the surface defined by $F = 0$ and $\phi = 0$. So, our partially gauge fixed theory has a constraint surface described by: four primary first-class constraints (m and m_a); one primary second-class constraint ($B' = \pi + 4 \langle p \rangle \sqrt{h} - 4p$); and five additional constraints (S, D^a, and ϕ). This theory is almost identical to Hamiltonian GR. Indeed, we observe that the surface in $\mathcal{P}_L$ defined by $\phi = 0$ and $B' = 0$ is isomorphic to $\mathcal{P}$. That is, we can identify this theory with an identical theory on $\mathcal{P}$ — namely, Hamiltonian GR.

4.2 Shape Dynamics

We will now derive shape dynamics by partially gauge fixing the linking theory constructed in the previous section. At each point in $\mathcal{S}$ we introduce the gauge-fixing constraint $\pi(x) = 0$ to obtain a partially gauge fixed theory — let us call it T_{L2}. To make the dynamics of T_{L2} consistent we demand that, for any $\eta(x)$, $X \langle \eta, \pi \rangle$ vanishes on the constraint surface. Now, recalling (4.9) we see that

$$X \langle \eta, \pi \rangle = \int \mathrm{d}^3 x \; \eta(x) \left(\frac{\delta \langle N, S' \rangle}{\delta \phi} + \frac{\delta \langle N_a, D'^a \rangle}{\delta \phi} \right) . \tag{4.10}$$

The second term in this integrand vanishes on the constraint surface. This is because $4p - 4 \langle p \rangle \sqrt{h} \approx \pi \approx 0$ such that, on the constraint surface, $\langle N_a, D'^a \rangle$ does not depend on $\phi(x)$ (see (**??**)). It remains to evaluate the first term in the integrand. With some effort one can show that, on the constraint surface,

$$\frac{\delta \langle N, S' \rangle}{\delta \phi} = \mathrm{e}^{6\hat{\phi}} \sqrt{h} \, \langle F \rangle - \sqrt{h} F, \tag{4.11}$$

where, on the constraint surface,

$$F = 8N\mathrm{e}^{6\hat{\phi}}\ {}^3R' + 2N\mathrm{e}^{6\hat{\phi}}\left\langle p\right\rangle^2 - 8\frac{1}{\sqrt{h}}\ {}^3\nabla^m\left(\mathrm{e}^{2\hat{\phi}}\sqrt{h}\ {}^3\nabla_m N\right). \tag{4.12}$$

So, combining Equations (4.10)—(4.12), we see that $X\left\langle \eta,\pi\right\rangle$ vanishes for all $\eta(x)$ iff

$$\left\langle\left({}^3R' + \frac{1}{4}\langle p'\rangle'^2 - {}^3\nabla'^2\right)N\right\rangle' = \left({}^3R' + \frac{1}{4}\langle p'\rangle'^2 - {}^3\nabla'^2\right)N, \tag{4.13}$$

where $\langle * \rangle'$ denotes the average with respect to h'_{ab} and ${}^3\nabla'^2$ is the Laplacian with respect to h'_{ab}. (For more details see Appendix C.) Since the LHS of (4.13) is a constant, this equation is just the lapse-fixing equation discussed by York (see [26, 27, 28]). This equation arises when one demands that the dynamics is consistent with the constant mean curvature (CMC) constraint and it is known that there exists some $\tilde{N}$ that satisfies (4.13). Given such a $\tilde{N}$ we can satisfy (4.10) by introducing an additional constraint: $C = N - \tilde{N}$. Finally, since we have a new constraint, we must continue the consistency algorithm and demand that $XC = [C, \langle\alpha, M\rangle] \approx 0$. Of course, this is satisfied iff $\alpha(x,t) = 0$ and this completes the consistency algorithm.

The constraint surface of T_{L2} is described by four primary first-class constraints at each point (M_a and $B = 4\langle p\rangle\sqrt{h} - 4p$), and seven other constraints (M, S', D^a, π and $C = N - \tilde{N}$). Notice that we have replaced D'^a with D^a since these are equal on the surface $B = 0, \pi = 0$. The canonical Hamiltonian is now $H = \left\langle \tilde{N}, S'\right\rangle + \langle N_a, D^a\rangle$, and the Hamiltonian vector fields are

$$X = \partial_t + [*, H] + [*, \langle\beta_a, M^a\rangle] + [*, \langle\gamma, B'\rangle]\,. \tag{4.14}$$

We see that introducing $\pi = 0$ has removed some of the dynamical gauge freedom from the linking theory whilst also leaving us with a new dynamical gauge freedom not present in Hamiltonian GR. Thus, we have successfully derived an equivalent theory to Hamiltonian GR that has different dynamical gauge freedoms.

Phase Space Reduction

T_{L2} is a theory on the extended phase space $\mathcal{P}_L$. However, we may be able to reduce T_{L2} to a theory on $\mathcal{P}$. The only constraint with ϕ-dependence in the theory is $S'(x)$. Thus, the possibility of phase space reduction depends on whether or not $\mathcal{P}_L|_{\pi=0,S'=0}$ is isomorphic to $\mathcal{P}$. To show the existence of this isomorphism it is sufficient to find some $\tilde{\phi}(x)$ such that $S' \approx 0$ on the surface $\phi = \tilde{\phi}$. If $\tilde{\phi}$ exists we can identify T_{L2} with the theory on $\mathcal{P}$ in which S' is no longer a constraint and the Hamiltonian is $H = \left\langle \tilde{N}, \tilde{S}\right\rangle + \langle N_a, D^a\rangle$, where $\tilde{S}(x)$ is $S'(x)$ with $\tilde{\phi}$ substituted for ϕ. This reduced theory, if it exists, is called shape dynamics.

Let us now investigate the existence of $\tilde{\phi}$. On the T_2 constraint surface we find that

$$\begin{aligned} S'(x) &= -\mathrm{e}^{2\hat{\phi}}\sqrt{h}\ {}^3R' + \tfrac{1}{\sqrt{h}}\mathrm{e}^{-6\hat{\phi}}\Big(p_{ab}p^{ab} - \tfrac{1}{2}p^2 + \tfrac{1}{3}\langle p\rangle p\sqrt{h} - \tfrac{1}{3}\mathrm{e}^{6\hat{\phi}}\langle p\rangle p\sqrt{h} \\ &\qquad\qquad - \tfrac{1}{6}\langle p\rangle^2 h + \tfrac{1}{3}\mathrm{e}^{6\hat{\phi}}\langle p\rangle^2 h - \tfrac{1}{6}\mathrm{e}^{12\hat{\phi}}\langle p\rangle^2 h\Big) \\ &\approx -\mathrm{e}^{2\hat{\phi}}\sqrt{h}\ {}^3R' - \tfrac{1}{6}\mathrm{e}^{6\hat{\phi}}\langle p\rangle^2\sqrt{h} + \tfrac{1}{\sqrt{h}}\mathrm{e}^{-6\hat{\phi}}\Big(p_{ab}p^{ab} - \tfrac{1}{2}p^2 + \tfrac{1}{3}\langle p\rangle p\sqrt{h} \\ &\qquad\qquad - \tfrac{1}{6}\langle p\rangle^2 h\Big) \\ &\approx -\mathrm{e}^{2\hat{\phi}}\sqrt{h}\ {}^3R' - \tfrac{1}{6}\mathrm{e}^{6\hat{\phi}}\langle p\rangle^2\sqrt{h} + \tfrac{1}{\sqrt{h}}\mathrm{e}^{-6\hat{\phi}}\Big(p_{ab}p^{ab} - \tfrac{2}{3}p\langle p\rangle\sqrt{h} + \tfrac{1}{3}\langle p\rangle^2 h\Big), \end{aligned} \tag{4.15}$$

where we have made repeated use of the fact that $p - \langle p \rangle \sqrt{h}$ vanishes on the T_2 constraint surface. If we define $\sigma_{ab} \equiv p_{ab} - \frac{1}{3} \langle p \rangle h_{ab} \sqrt{h}$ and $\Phi \equiv \mathrm{e}^{\tilde{\phi}}$, then (4.15) can be expressed as

$$S'(x) \approx \sqrt{h} \Phi \left(- {}^3R' \Phi + \frac{1}{h} \sigma_{ab} \sigma^{ab} \Phi^{-7} - \frac{1}{6} \langle p \rangle^2 \Phi^5 \right). \tag{4.16}$$

Setting the expression inside the parentheses equal to zero gives the Lichnerwicz-York (LY) equation which has unique solutions (on a boundary-free manifold) if σ_{ab} is traceless and divergence free. (See, in this regard, Ó Murchadha [29, 30].) Now, by our definition of σ_{ab} we have

$$\sigma^a{}_a = p - \langle p \rangle \sqrt{h}, \quad \text{and} \quad \nabla_a \sigma^{ab} = \nabla_a p^{ab} - \frac{1}{3} \langle p \rangle \nabla_a (h^{ab} \sqrt{h}) = \nabla_a p^{ab}. \tag{4.17}$$

Both of these expressions vanish on the T_2 constraint surface, and it follows by the properties of the LY equation that we can find $\tilde{\phi}$ such that $S'(x)$ vanishes on the constraint surface. Thus, the existence of shape dynamics (the phase space reduction of T_2) is guaranteed (for the boundary-free case that we are considering).

4.3 Gauge Freedom in Shape Dynamics

We summarise the previous section by observing that shape dynamics has a constraint set comprising, at each point, four primary first-class constraints (m_a and B), three secondary first-class constraints (D^a), and two second-class constraints (m and $n - \tilde{n}$). Moreover, the shape dynamics Hamiltonian is $H = \left\langle \tilde{n}, \tilde{S} \right\rangle + \langle n_a, D^a \rangle$. Now, we want to find a functional J that generates gauge transformations. Since J must be first class we will write it in the general form

$$J = \langle \zeta^a, m_a \rangle + \langle \eta, B \rangle + \left\langle \theta_b, D^b \right\rangle, \tag{4.18}$$

for arbitrary functions $\zeta^a(t, y), \eta(t, y)$ and $\theta_b(t, y)$. But, recalling our discussion in Section 2.3, J must satisfy

$$\frac{\partial J}{\partial t} + [J, H] = \text{combination of primary constraints.} \tag{4.19}$$

Evaluating the left-hand side gives

$$\frac{\partial J}{\partial t} + [J, H] = \left\langle \dot{\zeta}^a, m_a \right\rangle + \langle \dot{\eta}, B \rangle + \left\langle \dot{\theta}_b, D^b \right\rangle - \langle \zeta^a, D_a \rangle \tag{4.20}$$

from which we deduce that $\zeta^a = \dot{\theta}^a$. So, shape dynamics has two classes of gauge generators:

$$J_1 = \left\langle \dot{\theta}^a, m_a \right\rangle + \langle \theta^a, D_a \rangle, \quad \text{and} \quad J_2 = \langle \eta, B \rangle. \tag{4.21}$$

J_1 is familiar to us from our discussion of Hamiltonian GR; it generates 3-dimensional diffeomorphisms. (Recall the discussion of this generator in Chapter 2.) Thus, shape dynamics retains the diffeomorphism invariance of Hamiltonian GR. Now, J_2 generates transformations of the form

$$\delta h_{ab} = \frac{\delta J_2}{\delta p^{ab}} = (\langle 4\eta \rangle - 4\eta) h_{ab}, \tag{4.22}$$

where we have used the observation that

$$\begin{aligned}
\langle \eta, B \rangle &= \left\langle \langle 4\eta \rangle, p\sqrt{h} \right\rangle - \langle 4\eta, p \rangle \\
&\approx \left\langle \langle 4\eta \rangle, p\sqrt{h} \right\rangle - \langle 4\eta, p \rangle - \left\langle \langle 4\eta \rangle, \langle p \rangle \sqrt{h} - p \right\rangle \\
&= \langle \langle 4\eta \rangle - 4\eta, p \rangle .
\end{aligned} \tag{4.23}$$

The transformations (4.22) are conformal transformations of the 3-metric. In fact, they are also volume preserving since

$$\delta V = \delta \int \mathrm{d}^3x \, \sqrt{h} = \frac{3}{2} \int \mathrm{d}^3x \, \left(\langle 4\eta \rangle - 4\eta \right) \sqrt{h} = 0. \tag{4.24}$$

Thus, shape dynamics (on a closed manifold) has dynamical gauge invariance under volume-preserving conformal transformations.

Gauge Fixing

Shape dynamics and Hamiltonian GR have different gauge freedoms. However, by construction there must exist gauge fixings under which the two theories match. This common gauge fixing is called the dictionary theory. A simple way to obtain this theory is by gauge fixing the linking theory with $\{\phi = 0, \pi = 0\}$ and performing phase space reduction (the reader may wish to refer back to Figure 2.3 and Section 2.5). Adding $\phi = 0$ and $\pi = 0$ to T_L modifies the Hamiltonian vector field to become, after phase space reduction,

$$X = \partial_t + [*, H] + [*, \langle \beta_a, M^a \rangle], \tag{4.25}$$

where $H = \left\langle \tilde{N}, S \right\rangle + \langle N_a, D^a \rangle$. Moreover, the constraint surface of T_{dict} is defined by: M_a, $B = 4 \langle p \rangle \sqrt{h} - 4p$, M, S, D^a and $C = N - \tilde{N}$. T_{dict} is Hamiltonian GR in the constant mean curvature gauge and it can be easily obtained from Hamiltonian GR by adding the constraint $B = 0$ to Hamiltonian GR. Making this constraint consistent with the dynamics of Hamiltonian GR generates the lapse fixing constraint $N - \tilde{N}$ and reduces the Hamiltonian vector fields to (4.25). Similarly, T_{dict} can be obtained from shape dynamics by adding the constraint $S = 0$. This constraint does not commute with B'' and so the shape dynamics Hamiltonian vector field immediately reduces to (4.25).

A practical consequence is that if one has a constant-mean-curvature solution of Hamiltonian GR, one can also consider it to be the $S = 0$ gauge-fixing of a solution to shape dynamics. For instance, the Friedmann solution to Hamiltonian GR is automatically in the CMC gauge: the metric, extrinsic curvature, and conjugate momentum are all homogenous. Thus, we can see the Friedmann solution as a solution to shape dynamics — we are free to lift the $S = 0$ constraint and perform arbitrary, time-dependent volume-preserving conformal transformations (as generated by B'', which has reverted to being primary first-class). Note that the one true degree of freedom of the system, the total volume, remains non-gauge. The conformal freedom in shape dynamics has little practical use for analysing the Friedmann solution. In shape dynamics, one can choose any 3-metric conformally related to the constant-curvature 3-sphere. However, this has no physical consequences and it is much more convenient to consider a constant-curvature 3-sphere.

Chapter 5

Discussion

This report investigated the consequences of the problems with the Dirac approach to Hamiltonian gauge theories as espoused by Henneaux and Teitelboim [19] and many others. Original contributions to this discussion included extending Pons' discussion to include time-independent gauge transformations and proving appropriately weakened versions of claims found in Henneaux and Teitelboim. However, the most important result was the investigation of Hamiltonian GR. It was demonstrated that the lapse and shift are crucial to describing the gauge freedom of Hamiltonian GR in such a way that does not disagree with standard GR. In addition, shape dynamics was also investigated in this vein. However, the main findings of that analysis agreed with the literature.

Lagrange vs. Hamilton: A Pacifist's Perspective

The Dirac approach to Hamiltonian systems has caused some confusion in the literature over whether or not the Lagrangian and Hamiltonian approaches to dynamics are truly equivalent. For instance, in Section 1.2.2 of their book, Henneaux and Teitelboim give an example of a Lagrangian system that disagrees with the corresponding Dirac-derived Hamiltonian system. However, they disregard these differences because without Dirac they find that it is "not clear how to pass to quantum mechanics" [19]. Unfortunately, these disagreements between the Lagrangian and Hamiltonian formulations arise also for more important examples than those considered by Henneaux and Teitelboim. For instance, as pointed out Pitts, the Dirac conjecture does not hold for electromagnetism [31]. These disagreements are unnecessary and the approach taken in Chapter 2 does not result in any such discrepancies. In particular, it was shown explicitly in Appendix A that the Lagrangian and Hamiltonian treatments of electromagnetism agree when analysed without the Dirac approach.

These issues are of particular importance for general relativity which is a gauge theory with much more complicated gauge freedom than electromagnetism. In 1967, DeWitt published the Wheeler-DeWitt equation which amounts to quantising the constraints $S(x)$ and $D^a(x)$ and setting $\hat{S}\,|\Psi\rangle = 0$ and $\hat{D}^a\,|\Psi\rangle = 0$, where $|\Psi\rangle$ is an appropriate state vector [32]. This equation is cited by some authors (e.g. Anderson [1]) as a neat demonstration of the problem of time — unlike the Schödinger equation, the Wheeler-DeWitt equation does not determine any time evolution. However, time evolution and gauge transformations in Hamiltonian GR are not generated solely by S and D^a. The primary first-class constraints, m and m_a, also play a role. This observation certainly does not render quantisation trivial, but it would surely be useful to have a clear understanding of the classical theory and its gauge freedom before attempting quantisation. Indeed, in another well-known quantum gravity paper, Teitelboim investigates the quantisation of general relativity using the path integral approach [33]. In that paper Teitelboim takes $S(x)$ and $D^a(x)$ as gauge generators, but, as was shown in Chapter 3, this is subtly incorrect. Clearly, more work is needed to understand the implications of the gauge freedom in Hamiltonian GR for the problem of quantisation.

Shape Dynamics

Shape dynamics raises interesting questions about gauge freedom in Hamiltonian GR. In shape dynamics, the refoliation invariance of Hamiltonian GR is replaced by gauge invariance under spatial conformal transformations. However, it is unclear whether this reveals anything about the gauge freedom of Hamiltonian GR itself. To further this inquiry, it would be useful to establish a bijective correspondence between the conformal representatives in shape dynamics and the foliations of Hamiltonian GR. However, as recently noted by Gryb in a discussion of de Sitter spacetimes, this might not be possible [34].

Shape dynamics is constructed by considering an extended theory called the linking theory that can be partially gauge fixed to either shape dynamics or Hamiltonian GR. Moreover, both shape dynamics and Hamiltonian GR can be further gauge fixed to the same theory. It is in this way that shape dynamics and Hamiltonian GR are physically equivalent — they differ only in the gauge freedoms they exhibit. The construction of shape dynamics using a linking theory was successfully reproduced in Chapter 4 without using the Dirac approach. However, given the ubiquity of gauge theories in physics, it would be very interesting to investigate the linking formalism more generally. In the literature, the linking construction itself has only been studied using the language of Dirac and this seems to leave room for further analysis. For instance, as was pointed out in Chapter 2, Koslowski and Gomes [25] considered examples involving time-independent gauge freedom. However, it seems that the construction only has interesting applications to theories with dynamical gauge freedom (such as Hamiltonian GR).

As discussed in Chapter 4, shape dynamics is not helpful for the study of particularly simple solutions such as the Friedmann model in a closed universe. Nevertheless, the shape dynamics formalism may be of some use for studying inhomogenous models. In any shape dynamics solution, the time parameter is the York time — which is the time function that gives constant mean curvature (CMC) foliations in Hamiltonian GR. York time is considered by some researchers to be a useful choice of 'preferred time' in general relativity. However, in shape dynamics one is free to describe an inhomogenous spacetime by some conformally-related, but simpler choice of 3-metric. This may make it easier to study the cosmological implications of inhomogenous models.

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Appendix A

Hamiltonian Electromagnetism

The dynamics of the electromagnetic field $A^\mu(x)$, without sources, is described by the Lagrangian density $\mathcal{L} = -(1/4) \cdot F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The conjugate momentum relations are

$$\Pi_\mu = \frac{\partial \mathcal{L}}{\partial \partial_0 A^\mu} = -F_{0\mu}, \tag{A.1}$$

and so we have a single primary constraint, $\Pi_0(x)$, at each point x. The energy function is

$$E = \int \mathrm{d}^3x \left(\Pi_i \partial_0 A^i + \frac{1}{4} F_{\mu\nu}F^{\mu\nu} \right), \tag{A.2}$$

and we want to find a Hamiltonian that pulls back to this function under the Legendre transformation. We note that the conjugate momentum functions satisfy $\partial_0 A_i = -\Pi_i + \partial_i A_0$ and $(1/4)F_{\mu\nu}F^{\mu\nu} = (1/4)F_{mn}F^{mn} - (1/2)\Pi_m\Pi^m$. So, one possible Hamiltonian function is

$$H = \int \mathrm{d}^3x \ \mathcal{H} = \int \mathrm{d}^3x \ \left(\frac{1}{2}\Pi_m\Pi^m - A_0 \partial_i \Pi^i + \frac{1}{4} F_{mn}F^{mn} \right) \tag{A.3}$$

(where we have integrated by parts). The dynamics of the theory is generated by

$$X = \partial_t + [*, H] + [*, \langle \lambda(x,t), \Pi^0(x) \rangle], \tag{A.4}$$

for arbitrary $\lambda(x,t)$. Note that we now have infinitely many constraints each labelled by x and so we have passed to the continuum limit by defining

$$\langle f, g \rangle = \int \mathrm{d}^3x \ f(x)g(x). \tag{A.5}$$

Since these are functionals on our canonical variables, the Poisson bracket becomes, in the continuum limit,

$$[F[A^\mu(x), \Pi_\nu(x)], G[A^\mu(x), \Pi_\nu(x)]] = \int \mathrm{d}^3x \ \left(\frac{\delta F}{\delta A^\mu(x)} \frac{\delta G}{\delta \Pi_\mu(x)} - \frac{\delta G}{\delta A^\mu(x)} \frac{\delta F}{\delta \Pi_\mu(x)} \right). \tag{A.6}$$

The consistency algorithm leaves X unchanged and generates a new constraint, $\partial_i \Pi^i$.

A.1 The Dirac Conjecture for Electromagnetism

The Dirac conjecture is false for electromagnetism — both of the constraints are first class, but contrary to Dirac neither of them generates a gauge transformation. (The reader is referred to the recent discussion of Pitts [31].) Explicitly, $\Pi_0(x)$ generates transformations of the form

$$\delta A^\mu(x) = [A^\mu(x), \langle\lambda(y,t), \Pi_0(y)\rangle] = \delta^0_\mu \lambda(x,t). \tag{A.7}$$

This transformation alters the electric field by $\delta F_{0i} = \partial_0 \delta A_i - \partial_i \delta A_0 = -\partial_i \lambda(x,t)$. But the electric field is a physically observable quantity, and thus (A.7) is not a gauge transformation. Indeed, as pointed out by Pitts [31], this transformation violates Gauss' law — which makes it a very non-physical transformation, indeed. This was not noticed, apparently, by Sundermeyer in his presentation of electromagnetism [35].

Similarly, the other first class constraint, $\partial_i \Pi^i$, also generates a transformation that changes the electric field. In particular, it generates the transformation

$$\delta A_\mu(x) = \left[A_\mu, \left\langle \lambda, \partial_i \Pi^i \right\rangle\right] = -\delta^i_\mu \partial_i \lambda(x,t), \tag{A.8}$$

which changes the electric field by

$$\delta F_{0i} = -\partial_i \dot{\lambda}. \tag{A.9}$$

This also violates Gauss' law. Thus, neither of the first class constraints in canonical electromagnetism generates a gauge transformation.

A.2 Gauge Freedom in Electromagnetism

This analysis of gauge transformations resolves the problems that arose in our earlier discussion of electromagnetism. Recall that canonical electromagnetism has one primary first-class constraint, Π^0, and an additional first-class constraint, $\partial_i \Pi^i$. Now, suppose J generates a gauge transformation. Since J must be first class, it can be written in the form $J = \langle f, \Pi^0\rangle + \langle g, \partial_i \Pi^i\rangle$, for some arbitrary time-dependent functions on phase space $f(x,t)$ and $g(x,t)$. Now we apply the second condition. Observe that

$$\frac{\partial J}{\partial t} + [J, H] = \left\langle \dot{f}, \Pi^0 \right\rangle + \left\langle \dot{g}, \partial_i \Pi^i \right\rangle + \left\langle f, \partial_i \Pi^i \right\rangle. \tag{A.10}$$

This is a combination of the primary first-class constraints, $\Pi^0(x)$, if and only if $f(\mathbf{y},t) = -\dot{g}(\mathbf{y},t)$. Consequently, the most general gauge generator is

$$J = \left\langle g, \partial_i \Pi^i \right\rangle - \left\langle \dot{g}, \Pi^0 \right\rangle. \tag{A.11}$$

It is reassuring to check explicitly that this is a gauge generator. We find that J generates the transformation

$$\delta A_\mu(\mathbf{x}) = -\delta^i_\mu \partial_i g(\mathbf{x},t) - \delta^0_\mu \dot{g}(\mathbf{x},t) = \partial_\mu(-g(\mathbf{x},t)), \tag{A.12}$$

which is the gauge transformation with which we are familiar.

A.3 A Complete Gauge Fixing

To gauge fix the dynamical gauge freedom in electromagnetism we need to find constraints whose Poisson brackets with the primary first-class constraints, $\Pi^0(x,t)$, do not vanish. Let us choose $A^0(x,t) - F(x)$, for some time-independent function $F(x)$. Performing the consistency algorithm on this new constraint immediately removes the arbitrary function from the equations of motion. We are left with deterministic time evolution generated by $X_{(GF)} = \partial_t + [*, H]$, and initial data is now constrained by $\Pi^0(x,t)$, $\partial_i \Pi^i(x,t)$ and $A_0(x,t) - F(x)$. Let us now explicitly analyse any remaining gauge freedom of the theory. Recall that the gauge transformations are generated by (A.11). We note that

$$[A_0(x,t) - F(x), J] = 0 \quad \Rightarrow \quad \dot{g}(x,t) = 0. \tag{A.13}$$

So, the gauge generator (A.11) is restricted by the new constraint to take the form

$$J_{(GF)} = \int \mathrm{d}^3\mathbf{y} \left(\partial_i \Pi^i g(\mathbf{y})\right). \tag{A.14}$$

This new functional generates the (non-dynamical) transformations $\delta A_\mu(\mathbf{x}) = \delta^i_\mu \partial_i g(\mathbf{x})$. Thus, our theory still has non-dynamical gauge freedom in the initial data. This can be fixed by adding further constraints. The reader will already be familiar with several possibilities: for instance, the so-called 'Coulomb gauge' amounts to imposing the condition $\partial_i A^i = 0$. In fact, the Coulomb condition is not quite enough to completely gauge fix (A.14) since one is left with the transformations $\delta A_\mu(x) = \delta^i_\mu v_i$ for constant v_i. This residual freedom can be removed by choosing points at which the components A_i are constrained to vanish. A common choice is to demand that A_i goes to zero at infinity.

Appendix B

The Spherical Potential Problem

The state of this system can be described by a six-dimensional phase space $\mathcal{P}$ with coordinates (q_i, p_i) and bracket $[q_i, p_j] = \delta_{ij}$. The system has no primary constraints and time evolution is generated by the vector field $X = \partial_t + [*, H(\mathbf{q}, \mathbf{p})]$, where $H(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\mathbf{p}\cdot\mathbf{p} + V(\mathbf{q}\cdot\mathbf{q})$ is the canonical Hamiltonian. We construct a linking theory by adding the conjugate pair (ϕ, π) and the primary constraint $B = \pi$. Let us attempt to make z-axis rotations a dynamical gauge freedom of T_L by performing the transformation $\mathbf{q} \mapsto \mathbf{q}' = \mathbf{F}(\mathbf{q}, \phi) = R_3(\phi)\mathbf{q}$. The appropriate canonical transformation is generated by $K = p_i' F_i(\mathbf{q}, \phi) + \phi\pi'$ and after some computation we find that the transformed coordinates are

$$\begin{aligned} q_i' &= \delta_{i3}q_3 + (q_i - \delta_{i3}q_3)\cos(\phi) + \epsilon_{ij3}q_j \sin(\phi), \\ p_i' &= \delta_{i3}p_3 + (p_i - \delta_{i3}p_3)\cos(\phi) - \epsilon_{ij3}p_j \sin(\phi), \\ &\text{and} \quad \pi' = \pi - M(\mathbf{q}, \mathbf{p}, \phi), \end{aligned} \tag{B.1}$$

where $M(\mathbf{q}, \mathbf{p}, \phi) = (q_1 p_1 + q_2 p_2)\sin(2\phi) - \epsilon_{3ij} q_i p_j \cos(2\phi)$. Under this transformation the canonical Hamiltonian does not change $(H(\mathbf{q}', \mathbf{p}') = \frac{1}{2}\mathbf{p}'\cdot\mathbf{p}' + V(\mathbf{q}'\cdot\mathbf{q}') = \frac{1}{2}\mathbf{p}\cdot\mathbf{p} + V(\mathbf{q}\cdot\mathbf{q}))$ and our one primary constraint becomes $B' = \pi - M(\mathbf{q}, \mathbf{p}, \phi)$. Let us now investigate the dynamics of this theory. The primary constraint is no longer respected by the Hamiltonian vector field. We find

$$\dot{B} = XB = [H(\mathbf{q}, \mathbf{p}), M(\mathbf{q}, \mathbf{p}, \phi)] = 2V'(q^2)q_k \frac{\partial M}{\partial p_k} - p_k \frac{\partial M}{\partial q_k}. \tag{B.2}$$

This, in general, is non-zero. Following Dirac's consistency algorithm we introduce the additional constraint $E \equiv XB$ which ensures that $\dot{B} = 0$. However, E itself does not commute with B. An explicit calculation shows that demanding that $\dot{E} = 0$ constraints $\alpha(t)$ to be a phase space function that vanishes on $\phi = 0$. Altogether, demanding invariance under z-axis rotations has given us a theory with two constraints, B and E, in which the dynamics is generated by

$$X_L = \partial_t + [*, H(\mathbf{q}, \mathbf{p}) + F(\mathbf{q}, \mathbf{p}, \phi)B(\mathbf{q}, \mathbf{p}, \phi)]. \tag{B.3}$$

No arbitrary functions remain in X_L; that is, our system does not have any dynamical gauge freedom. Consequently, any theories obtained from T_L—whether by gauge fixing with $\phi = 0$ or $\pi = 0$—will not have z-axis rotations as a dynamical gauge freedom.

Appendix C

Shape Dynamics: Further Details

In this appendix we collect the details of the technical calculations presented in Chapter 4. Sections 1 and 2 calculate the effect of a volume preserving conformal transformation on the phase space variables and the constraint functions. Finally, in Section 3 we calculate a Poisson bracket which plays an important role in the gauge fixing of the linking theory.

C.1 The Transformation of the Extended Phase Space

We are considering the conformal transformation

$$(h_{ab}, p^{ab}; N, M; N^a, M_a) \mapsto (h'_{ab}, p'^{ab}; N', M'; N'^a, M'_a) \tag{C.1}$$

generated by

$$J = \int \mathrm{d}^3x \left(\mathrm{e}^{4\hat{\phi}} h_{ab} p'^{ab} + \phi\pi' + NM' + N^a M'_a \right). \tag{C.2}$$

In this section we will obtain explicit equations for the transformed coordinates under this transformation as quoted in (4.7). To begin we have

$$h'_{ab} = \frac{\delta J}{\delta p'^{ab}} = \mathrm{e}^{4\hat{\phi}} h_{ab}, \tag{C.3}$$

where, recalling (4.4), $\hat{\phi}$ is given by

$$\hat{\phi} = \phi - \frac{1}{6}\log\left\langle \mathrm{e}^{6\hat{\phi}} \right\rangle. \tag{C.4}$$

Now, to obtain p'^{ab} we consider

$$\begin{aligned} p^{ab} = \frac{\delta J}{\delta h_{ab}} &= \frac{\delta}{\delta h_{ab}} \left[\left\langle \mathrm{e}^{6\phi} \right\rangle^{-\frac{2}{3}} \int \mathrm{d}^3x \left(\mathrm{e}^{4\phi} h_{ab} p'^{ab} \right) \right] \\ &= \int \mathrm{d}^3x \left(\mathrm{e}^{4\phi} h_{ab} p'^{ab} \right) \frac{\delta}{\delta h_{ab}} \left\langle \mathrm{e}^{6\phi} \right\rangle^{-\frac{2}{3}} + \left\langle \mathrm{e}^{6\phi} \right\rangle^{-\frac{2}{3}} \frac{\delta}{\delta h_{ab}} \int \mathrm{d}^3x \left(\mathrm{e}^{4\phi} h_{ab} p'^{ab} \right), \end{aligned} \tag{C.5}$$

which follows by the product rule. Now, recalling the definition of $\langle * \rangle$, observe that

$$\begin{aligned}
\frac{\delta}{\delta h_{ab}} \left\langle \mathrm{e}^{6\hat{\phi}} \right\rangle^{-\frac{2}{3}} &= -\frac{2}{3} \left\langle \mathrm{e}^{6\hat{\phi}} \right\rangle^{-\frac{5}{3}} \frac{\delta}{\delta h_{ab}} \left[\frac{1}{V} \int \mathrm{d}^3 x \left(e^{6\phi} \sqrt{h} \right) \right] \\
&= -\frac{2}{3} \left\langle \mathrm{e}^{6\hat{\phi}} \right\rangle^{-\frac{5}{3}} \left[-\frac{1}{2V} \left\langle e^{6\phi} \right\rangle \sqrt{h} h^{ab} + \frac{1}{2V} \mathrm{e}^{6\phi} \sqrt{h} h^{ab} \right] \\
&= \frac{1}{3V} \left\langle e^{6\phi} \right\rangle^{-\frac{2}{3}} \left(1 - \mathrm{e}^{6\hat{\phi}} \right) \sqrt{h} h^{ab}.
\end{aligned} \tag{C.6}$$

The second term of (C.5) is easier to evaluate. We find

$$\frac{\delta}{\delta h_{ab}} \int \mathrm{d}^3 x \left(\mathrm{e}^{4\phi} h_{cd} p'^{cd} \right) = e^{4\phi} p'^{ab}. \tag{C.7}$$

Now we substitute (C.6) and (C.7) into (C.5) to obtain

$$\begin{aligned}
p^{ab} &= \frac{1}{3} \left(1 - \mathrm{e}^{6\hat{\phi}} \right) \sqrt{h} \left\langle h'_{cd} p'^{cd} \right\rangle h^{ab} + \mathrm{e}^{4\hat{\phi}} p'^{ab} \\
\Rightarrow \qquad p'^{ab} &= \mathrm{e}^{-4\hat{\phi}} \left[p^{ab} - \frac{1}{3} \left(1 - \mathrm{e}^{6\hat{\phi}} \right) \sqrt{h} \left\langle h'_{ab} p'^{ab} \right\rangle h^{ab} \right].
\end{aligned} \tag{C.8}$$

Finally, since

$$\left\langle 1 - \mathrm{e}^{6\hat{\phi}} \right\rangle = \langle 1 \rangle - \left\langle \mathrm{e}^{6\phi} \right\rangle^{-1} \left\langle \mathrm{e}^{6\phi} \right\rangle = 0, \tag{C.9}$$

we have $\left\langle h'_{ab} p'^{ab} \right\rangle = \left\langle h_{ab} p^{ab} \right\rangle$ and so (C.8) can be written as

$$p'^{ab} = \mathrm{e}^{-4\hat{\phi}} \left[p^{ab} - \frac{1}{3} \left(1 - \mathrm{e}^{6\hat{\phi}} \right) \sqrt{h} \left\langle p \right\rangle h^{ab} \right]. \tag{C.10}$$

This completes our discussion of h_{ab} and p^{ab}. Let us now consider the transformations of ϕ and π. By our construction of J, ϕ transforms as

$$\phi' = \frac{\delta J}{\delta \pi'} = \phi. \tag{C.11}$$

To obtain π' we consider

$$\pi = \frac{\delta J}{\delta \phi} = \pi' + \left\langle e^{6\phi} \right\rangle^{-\frac{2}{3}} \frac{\delta}{\delta \phi} \int \mathrm{d}^3 x \left(\mathrm{e}^{4\phi} h_{ab} p'^{ab} \right) + \int \mathrm{d}^3 x \left(\mathrm{e}^{4\phi} h_{ab} p'^{ab} \right) \frac{\delta}{\delta \phi} \left\langle \mathrm{e}^{6\phi} \right\rangle^{-\frac{2}{3}}, \tag{C.12}$$

where we have used the product rule. For the second term we find

$$\frac{\delta}{\delta \phi} \int \mathrm{d}^3 x \left(\mathrm{e}^{4\phi} h_{ab} p'^{ab} \right) = 4 \mathrm{e}^{4\phi} h_{ab} p'^{ab}, \tag{C.13}$$

and for the third term we find

$$\frac{\delta}{\delta \phi} \left\langle \mathrm{e}^{6\phi} \right\rangle^{-\frac{2}{3}} = -4 \mathrm{e}^{6\hat{\phi}} \sqrt{h} \left\langle \mathrm{e}^{4\hat{\phi}} h_{ab} p'^{ab} \right\rangle. \tag{C.14}$$

So, substituting (C.13) and (C.14) into (C.12) gives

$$\pi' = \pi - 4 h_{ab} \mathrm{e}^{4\hat{\phi}} p'^{ab} + 4 e^{6\hat{\phi}} \sqrt{h} \left\langle p \right\rangle. \tag{C.15}$$

At this point we now substitute (C.10) to find

$$\begin{aligned}\pi' &= \pi - 4h_{ab}\left[p^{ab} - \frac{1}{3}\left(1 - \mathrm{e}^{6\hat{\phi}}\right)\langle p\rangle h^{ab}\sqrt{h}\right] + 4e^{6\hat{\phi}}\sqrt{h}\,\langle p\rangle \\ &= \pi - 4p + 4\langle p\rangle\sqrt{h} - 4\mathrm{e}^{6\hat{\phi}}\langle p\rangle\sqrt{h} + 4e^{6\hat{\phi}}\sqrt{h}\,\langle p\rangle\,.\end{aligned} \tag{C.16}$$

That is,

$$\pi' = \pi - 4p + 4\langle p\rangle\sqrt{h}. \tag{C.17}$$

This concludes our discussion of ϕ and π. Finally, observe that the transformations of N, M, N^a, and M_a generated by J are all trivial, i.e. $N' = N$, $M' = M$, and so on.

C.2 The Transformation of the Constraints

We now calculate how the constraint functions change under the conformal transformation described in the previous section. Recall that, prior to the transformation, the linking theory has nine constraints at each point: $S(x), D^a(x), B(x), m(x)$ and $m_a(x)$. Recall that the constraint functions are

$$\begin{aligned}S(x) &= \sqrt{h}\;{}^3R - \frac{1}{\sqrt{h}}\left(p_{ab}p^{ab} - \frac{1}{2}p^2\right), \\ D^a(x) &= \sqrt{h}\;{}^3\nabla_b\left(\frac{1}{\sqrt{h}}p^{ab}\right), \\ &\text{and}\quad B(x) = \pi.\end{aligned} \tag{C.18}$$

Under the conformal transformation, $B(x)$ becomes

$$B'(x) = \pi - 4p + 4\langle p\rangle\sqrt{h}, \tag{C.19}$$

by (C.17). Similarly, $D^a(x)$ becomes

$$D'^a(x) = \mathrm{e}^{6\hat{\phi}}\sqrt{h}\;{}^3\nabla'_b\left\{\mathrm{e}^{-10\hat{\phi}}\frac{1}{\sqrt{h}}\left[p^{ab} - \frac{1}{3}\left(1 - \mathrm{e}^{6\hat{\phi}}\right)\sqrt{h}\,\langle p\rangle h^{ab}\right]\right\}, \tag{C.20}$$

where ${}^3\nabla'_b$ is the Levi-Civita connection with respect to the transformed metric h'_{ab}. In the text we are often concerned with the functional $\langle\beta_a, D'^a\rangle$ for some functions $\beta_a(x,t)$. Discarding boundary terms we can write $\langle\beta_a, D'^a\rangle$ as

$$\begin{aligned}\langle\beta_a, D'^a\rangle &= \int \mathrm{d}^3x\left[\beta_a\sqrt{h'}\;{}^3\nabla'_b\left(\frac{1}{\sqrt{h'}}p'^{ab}\right)\right] \\ &= -\int \mathrm{d}^3x\left[p'^{ab}\;{}^3\nabla'_b\beta_a\right] \\ &= -\frac{1}{2}\int \mathrm{d}^3x\left[p'^{ab}\left({}^3\nabla'_b\beta_a + {}^3\nabla'_a\beta_b\right)\right] \\ &= -\frac{1}{2}\int \mathrm{d}^3x\left[p'^{ab}\pounds_{\boldsymbol{\beta}}\,h'_{ab}\right].\end{aligned} \tag{C.21}$$

Now, expanding p'^{ab} and h'_{ab} we find

$$\begin{aligned} p'^{ab}\pounds_{\boldsymbol{\beta}}\left(\mathrm{e}^{4\hat{\phi}}h_{ab}\right) &= p'^{ab}\mathrm{e}^{4\hat{\phi}}\pounds_{\boldsymbol{\beta}}\,h_{ab} + 4\mathrm{e}^{4\hat{\phi}}p'^{ab}h_{ab}\pounds_{\boldsymbol{\beta}}\,\phi \\ &= \left[p^{ab} - \frac{1}{3}\left(1-\mathrm{e}^{6\hat{\phi}}\right)\sqrt{h}\,\langle p\rangle\,h^{ab}\right]\pounds_{\boldsymbol{\beta}}\,h_{ab} \\ &\qquad\qquad + 4\left[p - \left(1-\mathrm{e}^{6\hat{\phi}}\right)\sqrt{h}\,\langle p\rangle\right]\pounds_{\boldsymbol{\beta}}\,\phi \\ &= p^{ab}\pounds_{\boldsymbol{\beta}}\,h_{ab} + 4(p-\langle p\rangle\sqrt{h})\pounds_{\boldsymbol{\beta}}\,\phi - \frac{2}{3}\left(1-\mathrm{e}^{6\hat{\phi}}\right)\sqrt{h}\,\langle p\rangle\ {}^3\nabla_a\beta^a \\ &\qquad\qquad + 4\sqrt{h}\,\langle p\rangle\,\mathrm{e}^{6\hat{\phi}}\pounds_{\boldsymbol{\beta}}\,\phi. \end{aligned} \tag{C.22}$$

Substituting this back into (C.21) and discarding the boundary term gives us the final result

$$\langle\beta_a, D'^a\rangle = \int \mathrm{d}^3x\left[p^{ab}\pounds_{\boldsymbol{\beta}}h_{ab} + 4\left(p-\langle p\rangle\sqrt{h}\right)\pounds_{\boldsymbol{\beta}}\,\phi\right]. \tag{C.23}$$

It remains to calculate how $S(x)$ transforms. Under the conformal transformation $h_{ab} \to h'_{ab} = \mathrm{e}^{4\hat{\phi}}h_{ab}$ the Ricci scalar transforms as

$${}^3R \to {}^3R' = \mathrm{e}^{-4\hat{\phi}}\left({}^3R - 8\partial_a\partial^a\phi - 8\partial_a\phi\partial^a\phi\right). \tag{C.24}$$

The reader can find this calculation in Appendix G of Ref. [4], for instance. Substituting (C.24) and (C.10) into $S(x)$ gives

$$\begin{aligned} S'(x) &= \sqrt{h'}\ {}^3R' - \frac{1}{\sqrt{h'}}\left(p'_{ab}p'^{ab} - \frac{1}{2}p'^2\right) \\ &= \mathrm{e}^{2\hat{\phi}}\sqrt{h}\left({}^3R - 8\partial_a\partial^a\phi - 8\partial_a\phi\partial^a\phi\right) \\ &\qquad - \frac{\mathrm{e}^{-6\hat{\phi}}}{\sqrt{h}}\left[p_{ab}p^{ab} - \frac{2}{3}\left(1-\mathrm{e}^{6\hat{\phi}}\right)\langle p\rangle\sqrt{h}p + \frac{1}{3}\left(1-\mathrm{e}^{6\hat{\phi}}\right)^2\langle p\rangle^2 h\right] \\ &\qquad + \frac{\mathrm{e}^{-6\hat{\phi}}}{2\sqrt{h}}\left[p - \left(1-\mathrm{e}^{6\hat{\phi}}\right)\langle p\rangle\sqrt{h}\right]^2 \\ &= \mathrm{e}^{2\hat{\phi}}\sqrt{h}\left({}^3R - 8\partial_a\partial^a\phi - 8\partial_a\phi\partial^a\phi\right) \\ &\qquad - \frac{\mathrm{e}^{-6\hat{\phi}}}{\sqrt{h}}\left[p_{ab}p^{ab} - \frac{1}{2}p^2 + \frac{1}{3}\left(1-\mathrm{e}^{6\hat{\phi}}\right)\langle p\rangle\,p\sqrt{h} - \frac{1}{6}\left(1-\mathrm{e}^{6\hat{\phi}}\right)^2\langle p\rangle^2 h\right]. \end{aligned} \tag{C.25}$$

C.3 The Poisson Bracket of S' with π

As calculated above, the Hamiltonian constraint following the conformal transformation is

$$\begin{aligned} S'(x) &= \mathrm{e}^{2\hat{\phi}}\sqrt{h}\left({}^3R - 8\partial_a\partial^a\phi - 8\partial_a\phi\partial^a\phi\right) \\ &\qquad - \frac{\mathrm{e}^{-6\hat{\phi}}}{\sqrt{h}}\left[p_{ab}p^{ab} - \frac{1}{2}p^2 + \frac{1}{3}\left(1-\mathrm{e}^{6\hat{\phi}}\right)\langle p\rangle\,p\sqrt{h} - \frac{1}{6}\left(1-\mathrm{e}^{6\hat{\phi}}\right)^2\langle p\rangle^2 h\right]. \end{aligned} \tag{C.26}$$

In the text we need to calculate the bracket $[\pi, \langle N, S'\rangle]$. To find this we need to find the functional derivative of $\langle N, S'\rangle$ with respect to ϕ. This is the computation we explain in this section. To begin, observe that we can can break $S'(x)$ into four terms,

$$\begin{aligned} S'(x) = &\left\langle \mathrm{e}^{6\phi} \right\rangle^{-\frac{1}{3}} \left[-\mathrm{e}^{2\phi}\sqrt{h} \left({}^3R - 8\partial^2\phi - 8(\partial\phi)^2 \right) \right] \\ &+ \left\langle \mathrm{e}^{6\phi} \right\rangle \left[\frac{\mathrm{e}^{-6\phi}}{\sqrt{h}} \left(p_{ab}p^{ab} - \frac{1}{2}p^2 + \frac{1}{3}\langle p\rangle p\sqrt{h} - \frac{1}{6}\langle p\rangle^2 h \right) \right] \\ &+ \frac{1}{\sqrt{h}} \left(-\frac{1}{3}\langle p\rangle p\sqrt{h} + \frac{1}{3}\langle p\rangle^2 h \right) \\ &+ \left\langle \mathrm{e}^{6\phi} \right\rangle^{-1} \left(-\frac{1}{6}\mathrm{e}^{6\phi}\langle p\rangle^2 h \right). \end{aligned} \tag{C.27}$$

It is easiest to evaluate each of these four terms separately. For the first term we observe that

$$\begin{aligned} &\frac{\delta}{\delta\phi} \left\langle N, \left[-\mathrm{e}^{2\phi}\sqrt{h} \left({}^3R - 8\partial^2\phi - 8(\partial\phi)^2 \right) \right] \right\rangle \\ &\qquad = 2\mathrm{e}^{2\phi}\sqrt{h}N \left({}^3R - 8\partial^2\phi - 8(\partial\phi)^2 \right) - 8\nabla^m \left(\mathrm{e}^{2\phi}\sqrt{h}\nabla_m N \right). \end{aligned} \tag{C.28}$$

Moreover, recall from the previous sections that

$$\frac{\delta\left\langle \mathrm{e}^{6\phi} \right\rangle}{\delta\phi} = \frac{1}{V}6\mathrm{e}^{6\phi}\sqrt{h}. \tag{C.29}$$

Combining this with (C.28), and applying the product rule we find that

$$\begin{aligned} &\frac{\delta}{\delta\phi} \left\langle N, \left\langle \mathrm{e}^{6\phi} \right\rangle^{-\frac{1}{3}} \left[-\mathrm{e}^{2\phi}\sqrt{h} \left({}^3R - 8\partial^2\phi - 8(\partial\phi)^2 \right) \right] \right\rangle \\ &\qquad = \mathrm{e}^{6\hat{\phi}}\sqrt{h} \left\langle 2N\mathrm{e}^{6\hat{\phi}}\, {}^3R' \right\rangle - 2N\mathrm{e}^{6\hat{\phi}}\, {}^3R'\sqrt{h} \\ &\qquad\qquad + 8\nabla^m \left(\mathrm{e}^{2\hat{\phi}}\sqrt{h}\nabla_m N \right), \end{aligned} \tag{C.30}$$

where ${}^3R'$ is the Ricci scalar following the conformal transformation (as defined in (C.24)). This can be compactly expressed as

$$\frac{\delta}{\delta\phi} \left\langle N, \left\langle \mathrm{e}^{6\phi} \right\rangle^{-\frac{1}{3}} \left[-\mathrm{e}^{2\phi}\sqrt{h} \left({}^3R - 8\partial^2\phi - 8(\partial\phi)^2 \right) \right] \right\rangle = \mathrm{e}^{6\hat{\phi}}\sqrt{h}\langle A\rangle - A\sqrt{h}, \tag{C.31}$$

where we define

$$A = 2N\mathrm{e}^{6\hat{\phi}}\, {}^3R' - 8\frac{1}{\sqrt{h}}\nabla^m \left(\mathrm{e}^{2\hat{\phi}}\sqrt{h}\nabla_m N \right). \tag{C.32}$$

Observe that the second term of A becomes a boundary term in $\langle A\rangle$. This completes our calculation of the first term. The remaining terms in (C.27) are found in a similar fashion. The second term is

$$\frac{\delta}{\delta\phi} \left\langle N, \left\langle \mathrm{e}^{6\phi} \right\rangle \left[\frac{\mathrm{e}^{-6\phi}}{\sqrt{h}} \left(p_{ab}p^{ab} - \frac{1}{2}p^2 + \frac{1}{3}\langle p\rangle p\sqrt{h} - \frac{1}{6}\langle p\rangle^2 h \right) \right] \right\rangle = \mathrm{e}^{6\hat{\phi}}\sqrt{h}\langle B\rangle - B\sqrt{h}, \tag{C.33}$$

where we define

$$B = 6\mathrm{e}^{-6\hat{\phi}}\frac{N}{h}\left(p_{ab}p^{ab} - \frac{1}{2}p^2 + \frac{1}{3}\langle p\rangle p\sqrt{h} - \frac{1}{6}\langle p\rangle^2 h\right). \tag{C.34}$$

The third term vanishes since it does not depend on ϕ. Finally, the fourth term is

$$\frac{\delta}{\delta\phi}\left\langle N, \left\langle \mathrm{e}^{6\phi}\right\rangle^{-1}\left(-\frac{1}{6}\mathrm{e}^{6\phi}\langle p\rangle^2 h\right)\right\rangle = \mathrm{e}^{6\hat{\phi}}\sqrt{h}\,\langle C\rangle - C\sqrt{h}, \tag{C.35}$$

where

$$C = \mathrm{e}^{6\hat{\phi}}N\langle p\rangle^2 h. \tag{C.36}$$

Having performed these computations we can combinine (C.31)–(C.36) to obtain the final result:

$$\frac{\delta\langle N, S'\rangle}{\delta\phi} = \mathrm{e}^{6\hat{\phi}}\sqrt{h}\,\langle F\rangle - F\sqrt{h}, \tag{C.37}$$

where

$$\begin{aligned} F = {} & 2N\mathrm{e}^{6\hat{\phi}}\,{}^3R' - 8\frac{1}{\sqrt{h}}\nabla^m\left(\mathrm{e}^{2\hat{\phi}}\sqrt{h}\nabla_m N\right) + \mathrm{e}^{6\hat{\phi}}N\langle p\rangle^2 h \\ & + 6\mathrm{e}^{-6\hat{\phi}}\frac{N}{h}\left(p_{ab}p^{ab} - \frac{1}{2}p^2 + \frac{1}{3}\langle p\rangle p\sqrt{h} - \frac{1}{6}\langle p\rangle^2 h\right). \end{aligned} \tag{C.38}$$

This, however, is not the most convenient form of the result. Recalling the definition of p'^{ab} given in (C.8) we can write

$$\begin{aligned} F = {} & 2N\mathrm{e}^{6\hat{\phi}}\,{}^3R' - 8\frac{1}{\sqrt{h}}\nabla^m\left(\mathrm{e}^{2\hat{\phi}}\sqrt{h}\nabla_m N\right) + 6\mathrm{e}^{-6\hat{\phi}}\frac{N}{\sqrt{h}}\left(p'_{ab}p'^{ab} - \frac{1}{2}p'^2\right) \\ & + 2\mathrm{e}^{6\hat{\phi}}N\langle p\rangle^2\sqrt{h} + 2\frac{N}{\sqrt{h}}\langle p\rangle\left(p - \sqrt{h}\,\langle p\rangle\right). \end{aligned} \tag{C.39}$$

Finally, we observe that on the surface $S' = 0$, $B' = 0$ and $\pi = 0$, F simplifies to

$$F \approx 8N\mathrm{e}^{6\hat{\phi}}\,{}^3R' - 8\frac{1}{\sqrt{h}}\nabla^m\left(\mathrm{e}^{2\hat{\phi}}\sqrt{h}\nabla_m N\right) + 2\mathrm{e}^{6\hat{\phi}}N\langle p\rangle^2\sqrt{h}. \tag{C.40}$$

Bibliography

[1] E. Anderson. Problem of time in quantum gravity. *Annalen der Physik*, 524:757–786, 2012.

[2] A. Bernal and M. Sanchez. On smooth cauchy hypersurfaces and geroch's splitting theorem. *Communications in Mathematical Physics*, 243:461–470, 2003.

[3] R. Arnowitt, S. Deser, and C. Misner. *in: GRAVITATION: An Introduction to Current Research.* John Wiley & Sons, 2nd printing edition edition, 1963.

[4] Y. Fujii and K. Maeda. The scalar-tensor theory of gravitation. *Classical and Quantum Gravity*, 20:4503–4503, 2003.

[5] H. Gomes, S. Gryb, T. Koslowski, and F. Mercati. The gravity/CFT correspondence. *The European Physical Journal C*, 73, 2011. arXiv: 1105.0938.

[6] H. Gomes, S. Gryb, T. Koslowski, F. Mercati, and L. Smolin. Why gravity codes the renormalization of conformal field theories. 2013. arXiv: 1305.6315.

[7] T. Koslowski. Shape dynamics and effective field theory. 2013. arXiv: 1305.1487.

[8] J. Barbour. The definition of machs principle. *Foundations of Physics*, 40:1263–1284, 2010.

[9] Nick Huggett and Carl Hoefer. Absolute and relational theories of space and motion. 2009.

[10] O. Pooley. Substantivalist and relationalist approaches to spacetime. In R. Batterman, editor, *The Oxford Handbook of Philosophy of Physics.* Oxford University Press, 2013.

[11] E. Anderson. Relationalism. arXiv: 1205.1256.

[12] J. Barbour and B. Bertoth. Gravity and inertia in a machian framework. *Il Nuovo Cimento B*, 38:1–27, 1977.

[13] J. Barbour and B. Bertotti. Mach's principle and the structure of dynamical theories. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, 382:295–306, 1982.

[14] J. Barbour, T. Koslowski, and F. Mercati. A gravitational origin of the arrows of time. 2013. arXiv: 1310.5167.

[15] Julian Barbour, Tim Koslowski, and Flavio Mercati. Identification of a gravitational arrow of time. *Physical Review Letters (in press)*, 2014. arXiv: 1409.0917.

[16] T. Koslowski. Quantum inflation of classical shapes. 2014. arXiv: 1404.4815.

[17] H. Gomes, S. Gryb, and T. Koslowski. Einstein gravity as a 3d conformally invariant theory. *Classical and Quantum Gravity*, 28:045005, 2011.

[18] P. Dirac. *Lectures on Quantum Mechanics.* Yeshiva University, New York: Academic Press, 1967.

[19] M. Henneaux and C. Teitelboim. *Quantization of Gauge Systems*. Princeton University Press, Princeton, N.J., 1994.

[20] R. Wald. *General Relativity*. University Of Chicago Press, Chicago, 1984.

[21] L. Faddeev and R. Jackiw. Hamiltonian reduction of unconstrained and constrained systems. *Physical Review Letters*, 60:1692–1694, 1988.

[22] C. Batlle, J. Gomis, J. Pons, and N. Roman-Roy. Equivalence between the lagrangian and hamiltonian formalism for constrained systems. *Journal of Mathematical Physics*, 27:2953–2962, 1986.

[23] J. Pons and L. Shepley. Evolutionary laws, initial conditions and gauge fixing in constrained systems. *Classical and Quantum Gravity*, 12:1771, 1995.

[24] J. Pons. On dirac's incomplete analysis of gauge transformations. *Studies in History and Philosophy of Science*, 36:491–518, 2005.

[25] H. Gomes and T. Koslowski. The link between general relativity and shape dynamics. *Classical and Quantum Gravity*, 29:075009, 2012.

[26] J. York. Gravitational degrees of freedom and the initial-value problem. *Physical Review Letters*, 26:1656–1658, 1971.

[27] J. York. Role of conformal three-geometry in the dynamics of gravitation. *Physical Review Letters*, 28:1082–1085, 1972.

[28] J. York. Conformally invariant orthogonal decomposition of symmetric tensors on riemannian manifolds and the initialvalue problem of general relativity. *Journal of Mathematical Physics*, 14:456–464, 1973.

[29] N. O'Murchadha and J. York. Existence and uniqueness of solutions of the hamiltonian constraint of general relativity on compact manifolds. *Journal of Mathematical Physics*, 14:1551–1557, 1973.

[30] N. Ó Murchadha, C. Soo, and H. Yu. Intrinsic time gravity and the lichnerowicz-york equation. *Classical and Quantum Gravity*, 30:095016, 2013. arXiv: 1208.2525.

[31] J. Brian Pitts. A first class constraint generates not a gauge transformation, but a bad physical change. *Annals of Physics (in press)*, 2014.

[32] Bryce S. DeWitt. Quantum theory of gravity. i. the canonical theory. *Physical Review*, 160:1113–1148, August 1967.

[33] Claudio Teitelboim. Quantum mechanics of the gravitational field. *Physical Review D*, 25:3159–3179, June 1982.

[34] Sean Gryb. Observing shape in spacetime. 2014. arXiv: 1408.3989.

[35] K. Sundermeyer. The electromagmetic field. In *Constrained Dynamics*, Lecture Notes in Physics, pages 123–160. Springer Berlin Heidelberg, 1982.